Lecture Notes in College Algebra

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Preface

These notes were prepared for the College Algebra course taught at Temple University during the Summer 2026 term, of which I was the Instructor of Records. They have been written with the conviction that knowledge should be free, publicly available, and easily shared. Mathematics, like all intellectual endeavors, flourishes through open exchange and collective effort, and access to learning should not be limited by unnecessary barriers, too often raised just for the sake of profit.

The purpose of these notes is to provide a clear and accessible introduction to the material for the students attending the course, but they are intended to be freely distributed, reproduced, and improved upon, in the hope that they may be useful to students wishing to review the material, wherever they may be. There will be some mistakes, and you are encouraged to send me an email, so that I may correct them.

No claim to originality is made, and most of the time mathematical rigor is sacrificed in favor of hands-on understanding of the tools of the course. At the end of every section there are some exercises; the reader should attempt at solving all of them. Most of them are simple calculations, to measure the confidence of the reader with the use of the tools introduced in the previous section, while some of them, indicated by (☕), are more conceptual, and require a slightly deeper understanding of the material to solve.

Contents

Chapter 1

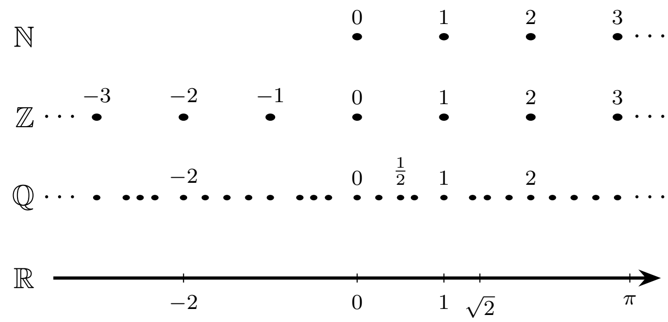
Functions, Graphs, and Polynomial Operations

1.1 Functions and Graphs

A great deal of mathematics, and almost all of its use in science and economics, rests on a single idea: one quantity depends on another. The amount you pay at a gas pump depends on how many gallons you take; the height of a thrown ball depends on how long it has been in the air; the area of a square depends on the length of its side. The concept of a *function* is the precise language we use to describe this kind of dependence, and it is the backbone of everything that follows in this course. We start by defining the main objects on which our study will take place: the various sets of numbers.

Number sets The most primitive collection is the set of *natural numbers* $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, the counting numbers you use to tally objects. Once you want subtraction to always be possible, for example to make sense of a debt, or of going below zero, you are forced to adjoin negative numbers, obtaining the *integers* $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. The next step is division: if you want to split a quantity evenly, you need fractions, and the *rational numbers* $\mathbb{Q}$ are all numbers expressible as a ratio p/q of two integers with $q \neq 0$[1]. Rationals cover an enormous range, but they leave gaps on the number line: lengths like $\sqrt{2}$ or quantities like π cannot be written as any such ratio. Filling in all those gaps, every point on the number line, gives the *real numbers* $\mathbb{R}$. Each step in this chain extends the previous set: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$, with every extension motivated by wanting some arithmetic operation to have no exceptions.

[1] This is important for the definition of the rationals to make sense at all.



Representation of the number sets as ordered sets.

Relations and Orders. A (binary) *relation* between two quantities, denoted by x and y, is simply any collection of ordered pairs (x, y).
For example, think of the order relation on natural numbers: saying that "3 is lesser than 5" is the same as saying that the ordered pair $(3, 5)$ is an element of the order relation.

Observe that we have implicitly defined an ordering of the number sets by representing them like in the figure above: between two elements, the rightmost one is the greater one. We will always think of numbers with an ordering. With that, comes a useful way to denote subsets of the real line, called *interval notation*.

Interval notation. An *interval* is the set of all real numbers lying between two endpoints a and b (with $a \leq b$). We write a square bracket to include an endpoint and a round bracket to exclude it:

$$[a, b] = \{x \in \mathbb{R} | \, a \leq x \leq b\}, \qquad (a, b) = \{x \in \mathbb{R} | \, a < x < b\},$$

the first being a *closed* interval and the second an *open* one; the mixed forms $[a, b)$ and $(a, b]$ are called *half-open*. To extend an interval indefinitely in one direction we use the symbols $-\infty$ and $+\infty$, which never count as endpoints and therefore always carry a round bracket: for instance $[a, +\infty) = \{x \in \mathbb{R} | \, x \geq a\}$ and $\mathbb{R} = (-\infty, +\infty)$.

For example, the set of real numbers x satisfying $-1 \leq x < 4$ is the half-open interval $[-1, 4)$: here -1 belongs to the set but 4 does not. As a second example, the inequality $x > 0$, describing the strictly positive numbers, is written $(0, +\infty)$, with a round bracket on 0 because the endpoint is excluded.

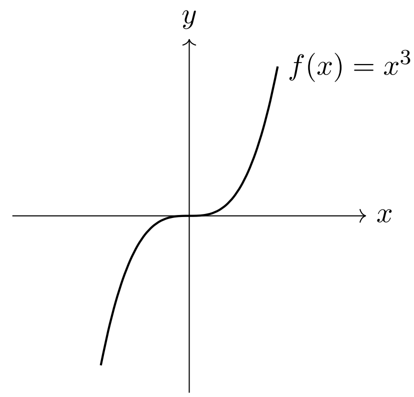
Functions. A *function* f is a relation that assigns to each input x in a set D (called the *domain*) exactly one output, denoted $f(x)$. The defining requirement is the phrase *exactly one*: an input is never allowed to produce more than one different outputs. The set R of all outputs that actually occur is called the *range* or *image*. We usually write

$$f : D \to R$$

to indicate that the domain of f is D, and the range is R. We read the previous line as "f, from D to R".

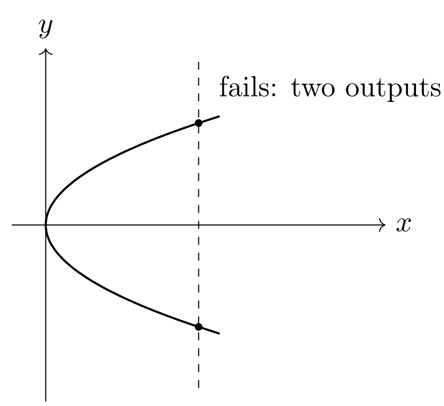
The notation $f(x)$ is read "f of x." It does not mean multiplication; it names the output that the function f produces from the input x. Evaluating a function means substituting a specific value for the input.
We will usually think of functions as "rules" defined on sets of numbers, although they do not need to be of that form. In this case, they can be associated with a geometric object, called the *graph*, which lives in a 2-dimensional plane.

Graphs. The *graph* of a function f is the set of all points $(x, f(x))$ in the coordinate plane. Because the output $f(x)$ is plotted directly above (or below) the input x on the horizontal axis, the requirement "exactly one output per input" has a vivid geometric meaning.



The graph of the cubing function $f(x) = x^3$.

The Vertical Line Test. A set of points in the plane is the graph of a function (of x) if and only if no vertical line meets it more than once. The reasoning is direct: a vertical line is the set of points with a fixed first coordinate $x = a$, so the points where it meets the graph are exactly the outputs assigned to the input a. Demanding "at most one such point" is precisely demanding "at most one output," which is the definition of a function.



The sideways parabola above (the relation $x = y^2$) is *not* a function of x: the dashed vertical line meets it twice, so the single input would have two outputs.

Example 1. Decide whether each relation is a function.

- $\{(1,2),(2,5),(3,5)\}$ is a function: every input $(1,2,3)$ has one output. (Two inputs sharing the output 5 is perfectly allowed.)
- $\{(1,2),(2,5),(1,7)\}$ is *not* a function: the input 1 is paired with both 2 and 7.

Example 2. Let $f(x) = x^2 - 3x$. Let us evaluate the function at the number 2, -1, and in a parametric expression $a + 1$:

$$f(2) = 2^2 - 3(2) = 4 - 6 = -2, \qquad f(-1) = (-1)^2 - 3(-1) = 1 + 3 = 4,$$

and, substituting the whole expression $a + 1$ for x,

$$f(a+1) = (a+1)^2 - 3(a+1) = a^2 + 2a + 1 - 3a - 3 = a^2 - a - 2.$$

1.1.1 Exercises

1. Decide whether $\{(0,1),(1,2),(2,3)\}$ is a function.
2. Decide whether $\{(-2,4),(0,0),(-2,1)\}$ is a function.
3. For $f(x) = 4x - 7$, compute $f(0)$, $f(3)$, and $f(-2)$.
4. For $g(x) = 2x^2 + 1$, compute $g(1)$, $g(-3)$, and $g(0)$.
5. For $h(x) = \dfrac{x+1}{x-2}$, compute $h(0)$ and $h(5)$.
6. For $f(x) = 5 - x^2$, compute $f(t)$ and $f(2t)$.
7. For $f(x) = x^2 - x$, simplify $f(a+1)$.
8. For $f(x) = 3x - 1$, simplify $f(x+h)$.
9. For $f(x) = x^2$, simplify the difference quotient $\dfrac{f(x+h) - f(x)}{h}$ (assume $h \neq 0$).
10. A taxi charges \$3.00 plus \$2.50 per mile. Write the cost C as a function of the number of miles m, and find $C(8)$.
11. Sketch a graph that passes the vertical line test and another that fails it.
12. Using the vertical line test, explain why a circle centered at the origin is not the graph of a function of x.
13. Can two different inputs of a function share the same output? Can one input have two outputs? Justify each answer using the definition.
14. If $f(2) = 7$ and $f(2) = k$, what must k equal? What property of functions forces this?
15. Suppose $f(x) = mx + b$ and $f(0) = 4$, $f(1) = 7$. Find m and b.
16. (☕) A graph is symmetric across the x-axis (meaning (x, y) is on the graph exactly when $(x, -y)$ is). Argue, without any algebra, that such a graph can be a function only if it lies entirely on the x-axis.

17. (☕) Explain why every nonvertical line is the graph of a function, but a vertical line is not.
18. (☕) The relation $x^2 + y^2 = 25$ is not a function. Without solving for y, identify a largest piece of it (a restriction of the outputs) that *is* a function, and describe it in words.

1.2 Finding the Domain and Range of Functions

When a function comes from a formula, not every number can legally be substituted. You cannot divide by zero, and within the real numbers you cannot take an even root of a negative quantity. A speedometer reading as a function of time only makes sense for times after the trip began. Determining which inputs are permissible, the *domain*, is therefore the first question to ask about any function.

Domain conventions. When a function defined on $\mathbb{R}$ is given only by a formula, its domain is understood to be the largest set of real numbers for which the formula produces a real output. Two rules account for nearly every case in this course:

- **Denominators may not be zero.** Exclude every input that makes a denominator 0.
- **Even roots require nonnegative radicands.** Inside a square root (or any even root), the expression must be ≥ 0. We will see this in more detail later.

A polynomial function, i.e., a function which can be expressed as a polynomial in the input, has no denominators and no roots (as in radical expressions), so its domain is always all real numbers; in interval notation, $(-\infty, \infty)$.

Finding the range. The range is the collection of outputs of the function, given the set of inputs: there is no general recipe for determining the range of a given function, and it is often read most easily from a graph, or deduced from the structure of a formula: for instance, any squared number is ≥ 0, and similarly a square root produces only nonnegative values.

Equality of functions. We say that two functions f and g, defined on the same domain D, are equal if $f(x) = g(x)$ for every element x of D. This principle is called *extensionality*, and it is an instance of a more general notion of equality between sets. In a sentence, "two functions are equal if they produce the same output from each input".

Example. Find the domain of $f(x) = \dfrac{x+4}{x^2-9}$.

The denominator factors as $x^2 - 9 = (x-3)(x+3)$, which is zero when $x = 3$ or $x = -3$. These are the only forbidden inputs, so the domain is

$$(-\infty, -3) \cup (-3, 3) \cup (3, \infty).$$

1.2.1 Exercises

1. Find the domain of $f(x) = 3x - 1$.
2. Find the domain of $f(x) = x^2 + 5x - 2$.
3. Find the domain of $f(x) = \frac{1}{x}$.
4. Find the domain of $f(x) = \frac{x}{x-7}$.
5. Find the domain of $f(x) = \frac{2x}{x+5}$.
6. Find the domain of $f(x) = \sqrt{x+6}$.
7. Find the domain of $f(x) = \sqrt{5-x}$.
8. Find the domain of $f(x) = \frac{4}{x^2-16}$.
9. Find the domain of $f(x) = \frac{x-1}{x^2+x-12}$.
10. Find the domain of $f(x) = \frac{\sqrt{x}}{x-4}$.
11. Find the domain of $f(x) = \frac{1}{\sqrt{x-3}}$ (note the strict requirement here).
12. Find the range of $f(x) = x^2 + 1$.
13. Find the range of $f(x) = -\sqrt{x}$.
14. Find the range of $f(x) = |x| - 2$.
15. Find the domain and range of $f(x) = \sqrt{9-x^2}$.
16. (☕) In Exercise 11 the denominator contains a square root. Explain why the domain there is $(3, \infty)$ rather than $[3, \infty)$, combining both domain rules in your answer.
17. (☕) Two students simplify $f(x) = \frac{x^2-4}{x-2}$ to $x+2$. One says the domain is all reals; the other says $x \neq 2$. Who is right, and why does simplifying not change the domain?
18. (☕) Without computing, explain why the range of $f(x) = \frac{1}{x^2+1}$ is contained in $(0, 1]$.

1.3 Systems of Equations in Two Unknowns

Many situations impose two or more conditions at once. For example, a company breaks even when the revenue equals the costs, a market settles where the supply price equals the demand price, and many more examples come to mind. Each of these conditions is an equation in two unknowns, and we seek values that satisfy them *simultaneously*. Such a collection of equations is called a *system*. We start by studying the simplest case, the one in which the system is made of two linear equations, i.e., polynomial equations of degree 1, with two unknowns.

Solutions. A solution of a system of equations in x and y is an ordered pair of numbers (x_0, y_0) that makes *every* equation true. As many things, solving a linear system in two unknowns has a concrete geometrical meaning: each linear equation represents a line, and a solution is a point lying on *both* lines, an intersection point. This allows us to distinguish three cases, following the possible intersections of lines in a plane.

Three possibilities. Two lines in the plane can intersect in exactly one point (in this case, the system is going to have a unique solution), be parallel and never meet (in this case, the system has no solution), or be the very same line (in this case, the system is going to have infinitely many solutions, corresponding to the line itself). There is no other possibility, which is why a linear system can never have, say, exactly two solutions.

Two algebraic methods.

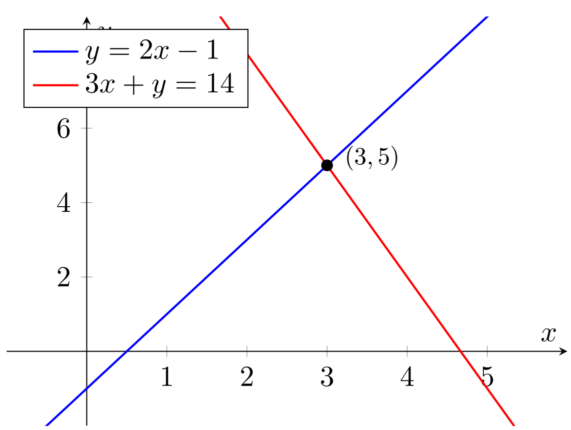
- **Substitution.** Solve one equation for one variable and substitute into the other, reducing the problem to a single equation in a single variable.
- **Elimination.** Add a suitable multiple of one equation to the other so that one variable cancels.

Example 1 (Substitution). Solve

$$\begin{cases} y = 2x - 1, \\ 3x + y = 14. \end{cases}$$

Substitute $y = 2x - 1$ into the second equation: $3x + (2x - 1) = 14$, so $5x = 15$ and $x = 3$. Then $y = 2(3) - 1 = 5$. The solution is $(3, 5)$, and you should check that it satisfies both equations.

Graphically, the solution is the point where the two lines cross:



Example 2 (Elimination). Solve

$$\begin{cases} 2x + 3y = 7, \\ 4x - 3y = 5. \end{cases}$$

Adding the two equations eliminates y: $6x = 12$, so $x = 2$. Substituting back, $2(2) + 3y = 7$ gives $3y = 3$, so $y = 1$. The solution is $(2, 1)$.

1.3.1 Exercises

1. Solve by substitution: $y = x + 2,\ 2x + y = 11$.
2. Solve by substitution: $x = 3y - 1,\ x + 2y = 9$.
3. Solve by elimination: $x + y = 6,\ x - y = 2$.
4. Solve by elimination: $3x + 2y = 12,\ x - 2y = 4$.
5. Solve: $2x + y = 5,\ 4x + 2y = 10$.
6. Solve: $x - y = 3,\ 2x - 2y = 7$.
7. Solve: $5x - 3y = 1,\ 2x + y = 8$.
8. Solve: $\frac{1}{2}x + y = 3,\ x - y = 0$.
9. Solve: $3x + 4y = 10,\ 2x - y = 7$.
10. Two numbers add to 30, and one is 4 more than the other. Find them.
11. I'm buying some large coffees, which cost \$2.50 each, and some bagels, at \$4.00 each. I spent \$33.00 across 10 items. How many bagels were bought? How many coffees?
12. A boat travels 30 mi downstream in 2 h and 30 mi upstream in 3 h. Find the boat's speed and the current's speed.
13. Solve: $\frac{3}{10}x + \frac{1}{5}y = \frac{13}{10},\ \frac{1}{2}x - \frac{1}{10}y = \frac{4}{5}$.
14. For what value of k does $2x + y = 5,\ 4x + 2y = k$ have infinitely many solutions?
15. For what value of k does the same system have no solution?
16. (☕) Explain, using the geometry of lines, why a system of two linear equations cannot have exactly two solutions.
17. (☕) Without solving, decide how many solutions $3x - y = 4,\ -6x + 2y = 1$ has, by comparing slopes.
18. (☕) If elimination leads to the false statement $0 = 5$, what does that tell you about the lines, and how many solutions are there?

1.4 Multiplication of Polynomials

We know that polynomials are formal expressions involving powers of a symbol, called the *variable*, and numbers. Adding polynomials is done by adding like terms, and so it is not particularly interesting. Multiplication is a bit more subtle, but it can be made much simpler using the distributive property of products with respect to sums.

Distributive property. Multiplication of polynomials is governed entirely by the distributive property: given (for the sake of simplicity) three polynomials a, b and c, we have

$$a(b+c) = ab + ac.$$

Observe that a, b and c here are *polynomials*, and so they may consist of many terms. Let us see the distributive property at work explicitly by multiplying (x^2+2x-1) by $(3x-4)$. We first distribute each term of the first factor across the second, then expand each of the resulting products, and finally collect like terms:

$$\begin{aligned}(x^2+2x-1)(3x-4) &= x^2(3x-4) + 2x(3x-4) - 1(3x-4)\\ &= (3x^3-4x^2) + (6x^2-8x) + (-3x+4)\\ &= 3x^3 - 4x^2 + 6x^2 - 8x - 3x + 4\\ &= 3x^3 + 2x^2 - 11x + 4.\end{aligned}$$

Special products. A few products occur so often that recognizing them saves work:

$$(a+b)(a-b) = a^2 - b^2, \qquad (a+b)^2 = a^2 + 2ab + b^2, \qquad (a-b)^2 = a^2 - 2ab + b^2.$$

Each is just the distributive law carried out once and remembered. The middle term $2ab$ in a perfect square is the part students most often drop[2]; note that $(a+b)^2 \neq a^2+b^2$. These three products are called "sum times difference", "square of a sum" and "square of a difference" for (one would hope) obvious reasons.

Example. Expand $(2x-3)(x+4)$.

$$(2x-3)(x+4) = 2x\cdot x + 2x\cdot 4 - 3\cdot x - 3\cdot 4 = 2x^2 + 8x - 3x - 12 = 2x^2 + 5x - 12.$$

Example 2. Expand $(3x-5)^2$ using the perfect-square pattern with $a = 3x$, $b = 5$:

$$(3x-5)^2 = (3x)^2 - 2(3x)(5) + 5^2 = 9x^2 - 30x + 25.$$

1.4.1 Exercises

1. Expand $3x(2x-5)$.
2. Expand $-2x^2(x^2-4x+1)$.
3. Expand $(x+6)(x+2)$.
4. Expand $(x-7)(x+3)$.
5. Expand $(2x+1)(3x-4)$.
6. Expand $(5x-2)(5x+2)$.

[2] It is called the "sophomore's dream" exactly for this reason.

7. Expand $(x+9)(x-9)$.
8. Expand $(x+4)^2$.
9. Expand $(2x-7)^2$.
10. Expand $(x+1)(x^2-x+1)$.
11. Expand $(x-2)(x^2+2x+4)$.
12. Expand $(2x+3)(x^2-x+5)$.
13. Expand $(x+1)(x+2)(x+3)$.
14. Find the coefficient of x^2 in $(x-3)(2x^2+x-5)$.
15. Write the area of a rectangle with sides $2x+1$ and $3x+4$ as a polynomial.
16. (☕) Without expanding fully, find the constant term and the leading term of $(3x-2)(x+5)(2x-1)$, and explain how you know them without doing all the work.
17. (☕) Explain why $(a+b)^2 = a^2+2ab+b^2$ by interpreting both sides as the area of a square of side $a+b$.
18. (☕) Use the difference-of-squares pattern to compute 103×97 mentally, and explain the trick.

1.5 Long Division of Polynomials

Dividing one polynomial by another mirrors the long division of whole numbers you learned years ago, and being able to do long division is one of the most interesting properties that an algebraic structure can have. Polynomial division is the gateway to factoring higher-degree polynomials and to analyzing rational functions later in the course, and it is the only real failsafe method for factoring polynomials: if you were to only learn one method of factoring, it should undoubtedly be long division.

The Division Algorithm. Given a polynomial dividend $p(x)$ and a nonzero divisor $d(x)$, there exist unique polynomials, the *quotient* $q(x)$ and the *remainder* $r(x)$, such that

$$p(x) = d(x)\,q(x) + r(x), \qquad \text{where } \deg r(x) < \deg d(x).$$

The condition on the degree of the remainder is what tells you when to stop: you keep dividing until what remains is of lower degree than the divisor, exactly as in arithmetic you stop when the remainder is smaller than the divisor.

Procedure. Arrange both polynomials in descending powers, inserting zero coefficients for any missing powers. Divide the leading term of the current dividend by the leading term of the divisor, multiply the divisor by that result, subtract, and repeat, until the remainder has low enough degree to force you to stop.

Example 1. Divide $2x^3 + 3x^2 - x + 5$ by $x + 2$.

$$\begin{array}{r} 2x^2 - x + 1 \\ x + 2\,\overline{\smash{)}\,2x^3 + 3x^2 - x + 5} \end{array}$$

Carrying out the steps: $2x^3 \div x = 2x^2$; subtracting $2x^2(x+2) = 2x^3 + 4x^2$ leaves $-x^2 - x + 5$. Next $-x^2 \div x = -x$; subtracting $-x(x+2) = -x^2 - 2x$ leaves $x + 5$. Finally $x \div x = 1$; subtracting $1 \cdot (x+2) = x + 2$ leaves the remainder 3. Thus

$$2x^3 + 3x^2 - x + 5 = (x+2)(2x^2 - x + 1) + 3.$$

Example 2. Divide $x^3 - 7$ by $x - 1$. Here several powers are missing, so we insert zero coefficients and write the dividend as $x^3 + 0x^2 + 0x - 7$.

$$\begin{array}{r} x^2 + x + 1 \\ x - 1\,\overline{\smash{)}\,x^3 + 0x^2 + 0x - 7} \end{array}$$

Carrying out the steps: $x^3 \div x = x^2$; subtracting $x^2(x-1) = x^3 - x^2$ leaves $x^2 + 0x - 7$. Next $x^2 \div x = x$; subtracting $x(x-1) = x^2 - x$ leaves $x - 7$. Finally $x \div x = 1$; subtracting $1 \cdot (x-1) = x - 1$ leaves the remainder -6. Thus

$$x^3 - 7 = (x-1)(x^2 + x + 1) - 6.$$

Factoring via Long Division. We can use long division to factor polynomials of arbitrary degree, if we are able to somehow find its roots. The strategy is to find one root at a time, and each time we discover a root $x = c$, the polynomial $x - c$ divides our polynomial exactly (with remainder 0), and dividing peels off that factor and hands us a quotient of one lower degree. We can then repeat until what is left is a degree one polynomial, which needs no factoring. Two tools make this possible: the *rational roots test*, which tells us where to look for roots, and *Ruffini's theorem*, which tells us that the procedure described above is mathematically sound.

The Factor Theorem and Ruffini's theorem. The link between roots and factors is Ruffini's Theorem[3]: $x - c$ is a factor of $p(x)$ if and only if $p(c) = 0$. This is an immediate consequence of the Remainder Theorem, which says that the remainder of dividing $p(x)$ by $x - c$ equals $p(c)$; if that remainder is zero, then $p(x) = (x - c)q(x)$ exactly. So the plan is: find a root c, often by guessing and checking, aided by the rational roots test, and then use long division to divide out $x - c$. We then iterate this procedure until we find a small enough remainder.

The Rational Roots Test. Of course, we cannot guess roots blindly. If

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

[3] Named after the Italian mathematician and physician Paolo Ruffini (1765-1822).

has integer coefficients, then every *rational* root, written in lowest terms as $\frac{p}{q}$, must have

$$p \mid a_0 \quad (p \text{ divides the constant term}) \qquad \text{and} \qquad q \mid a_n \quad (q \text{ divides the leading coefficient}).$$

This gives a finite list of candidates to try. When the leading coefficient is 1, the candidates are simply the (positive and negative) integer divisors of the constant term, which is the common case. The test does not promise that any of these candidates is actually a root, only that no other rational number can be; but in practice it usually turns up the roots we need.

Example 3. Factor $p(x) = x^3 - 4x^2 + x + 6$ completely.

The constant term is 6 and the leading coefficient is 1, so the rational root candidates are the divisors of 6: $\pm1, \pm2, \pm3, \pm6$. Testing them:

$$p(1) = 1 - 4 + 1 + 6 = 4 \neq 0, \qquad p(-1) = -1 - 4 - 1 + 6 = 0.$$

So $x = -1$ is a root and $x + 1$ is a factor. Dividing $p(x)$ by $x + 1$ by long division:

$$\begin{array}{r} x^2 - 5x + 6 \\ x + 1\,\overline{)\, x^3 - 4x^2 + x + 6} \end{array}$$

Carrying out the steps: $x^3 \div x = x^2$; subtracting $x^2(x + 1) = x^3 + x^2$ leaves $-5x^2 + x + 6$. Next $-5x^2 \div x = -5x$; subtracting $-5x(x+1) = -5x^2 - 5x$ leaves $6x+6$. Finally $6x \div x = 6$; subtracting $6(x + 1) = 6x + 6$ leaves the remainder 0. The quotient is $x^2 - 5x + 6$, so

$$x^3 - 4x^2 + x + 6 = (x + 1)(x^2 - 5x + 6).$$

Repeating this procedure (do it!), we get $x^2 - 5x + 6 = (x - 2)(x - 3)$, so the complete factorization is

$$x^3 - 4x^2 + x + 6 = (x + 1)(x - 2)(x - 3),$$

with roots $x = -1, 2, 3$.

Example 4. Factor $p(x) = 2x^3 - 3x^2 - 3x + 2$ completely.

Now $a_0 = 2$ and $a_n = 2$. The numerators p divide 2 (so $p \in \{\pm1, \pm2\}$) and the denominators q divide 2 (so $q \in \{1, 2\}$). The candidate roots $\frac{p}{q}$ are therefore

$$\pm1, \quad \pm2, \quad \pm\tfrac{1}{2}.$$

Testing $x = 2$:

$$p(2) = 2(8) - 3(4) - 3(2) + 2 = 16 - 12 - 6 + 2 = 0,$$

so $x - 2$ is a factor. Dividing $p(x)$ by $x - 2$ by long division:

$$\begin{array}{r} 2x^2 + x - 1 \\ x - 2\,\overline{)\, 2x^3 - 3x^2 - 3x + 2} \end{array}$$

Carrying out the steps: $2x^3 \div x = 2x^2$; subtracting $2x^2(x-2) = 2x^3 - 4x^2$ leaves $x^2 - 3x + 2$. Next $x^2 \div x = x$; subtracting $x(x-2) = x^2 - 2x$ leaves $-x + 2$. Finally $-x \div x = -1$; subtracting $-1 \cdot (x-2) = -x + 2$ leaves the remainder 0. The quotient is $2x^2 + x - 1$, so

$$2x^3 - 3x^2 - 3x + 2 = (x-2)(2x^2 + x - 1).$$

Factoring the quadratic in the same way, $2x^2+x-1 = (2x-1)(x+1)$, gives the complete factorization

$$2x^3 - 3x^2 - 3x + 2 = (x-2)(2x-1)(x+1),$$

with roots $x = 2,\ \frac{1}{2},\ -1$. Notice that the root $\frac{1}{2}$ is exactly one of the noninteger candidates the rational roots test warned us to check.

1.5.1 Exercises

1. Divide $x^2 + 5x + 6$ by $x + 2$.
2. Divide $x^2 - 2x - 15$ by $x - 5$.
3. Divide $2x^2 + 7x + 3$ by $x + 3$.
4. Divide $x^3 - 1$ by $x - 1$.
5. Divide $x^3 + 8$ by $x + 2$.
6. Divide $x^3 + 2x^2 - 5x - 6$ by $x - 2$.
7. Divide $2x^3 - 3x^2 + 4x - 1$ by $x + 1$.
8. Divide $x^4 - 16$ by $x - 2$.
9. Divide $3x^3 + 0x^2 + 0x - 24$ by $x - 2$.
10. Divide $x^3 - 7x + 6$ by $x + 3$.
11. Divide $x^4 - 5x^2 + 4$ by $x - 1$.
12. Divide $6x^2 + x - 2$ by $2x - 1$.
13. Find the remainder when $x^3 + 2x^2 - x + 4$ is divided by $x - 1$.
14. Determine whether $x + 4$ is a factor of $x^3 + 3x^2 - 4x - 12$.
15. Find the value of k so that $x - 2$ divides $x^3 - 3x^2 + kx - 2$ evenly.
16. () The Remainder Theorem says the remainder on dividing $p(x)$ by $x - c$ equals $p(c)$. Use it to find the remainder of Exercise 13 without dividing, and explain why it agrees.
17. () Explain, using the Division Algorithm, why the remainder on dividing by a linear polynomial $x - c$ must be a constant.
18. () If dividing $p(x)$ by $x - c$ gives remainder 0, what can you conclude about c and about the factorization of $p(x)$?

1.6 Factoring Techniques

Factoring is, in short, multiplication run in reverse: instead of combining factors into a single polynomial, we recover the factors from the polynomial. The reason why this works is the *zero-product property*, which says that a product is zero exactly when one of its factors is zero, which helps us turn a high-degree equation into smaller, nimbler ones.
How do we go about factoring polynomials without having to do long division? There are several ways, depending on how the given polynomial looks.

Greatest common factor. Always begin by factoring out the greatest common factor (GCF) of all terms. For example, $6x^3 - 9x^2 = 3x^2(2x - 3)$. The greatest common factor is defined as the highest-degree monomial that divides all of the terms of the polynomial, and it is unique.

Factoring by grouping. A four-term polynomial can sometimes be grouped into pairs sharing a common factor: $x^3 + 2x^2 + 3x + 6 = x^2(x + 2) + 3(x + 2) = (x + 2)(x^2 + 3)$.

Trinomials. To factor $x^2 + sx + p$, look for two numbers whose product is p and whose sum is s. For a leading coefficient other than 1, as in $ax^2 + sx + c$, find two numbers multiplying to ac and adding to s, then split the middle term and group.

Special forms. Recognizing a pattern is faster than any general method:

$$a^2 - b^2 = (a + b)(a - b), \quad a^2 + 2ab + b^2 = (a + b)^2, \quad a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2).$$

Observe that the equalities above are simply the ones we saw in Section 1.4, written conversely. Since an equality is *by definition* symmetrical, there is really no new information here. Note that a *sum* of squares $a^2 + b^2$ does not factor over the real numbers. (☕) Why?

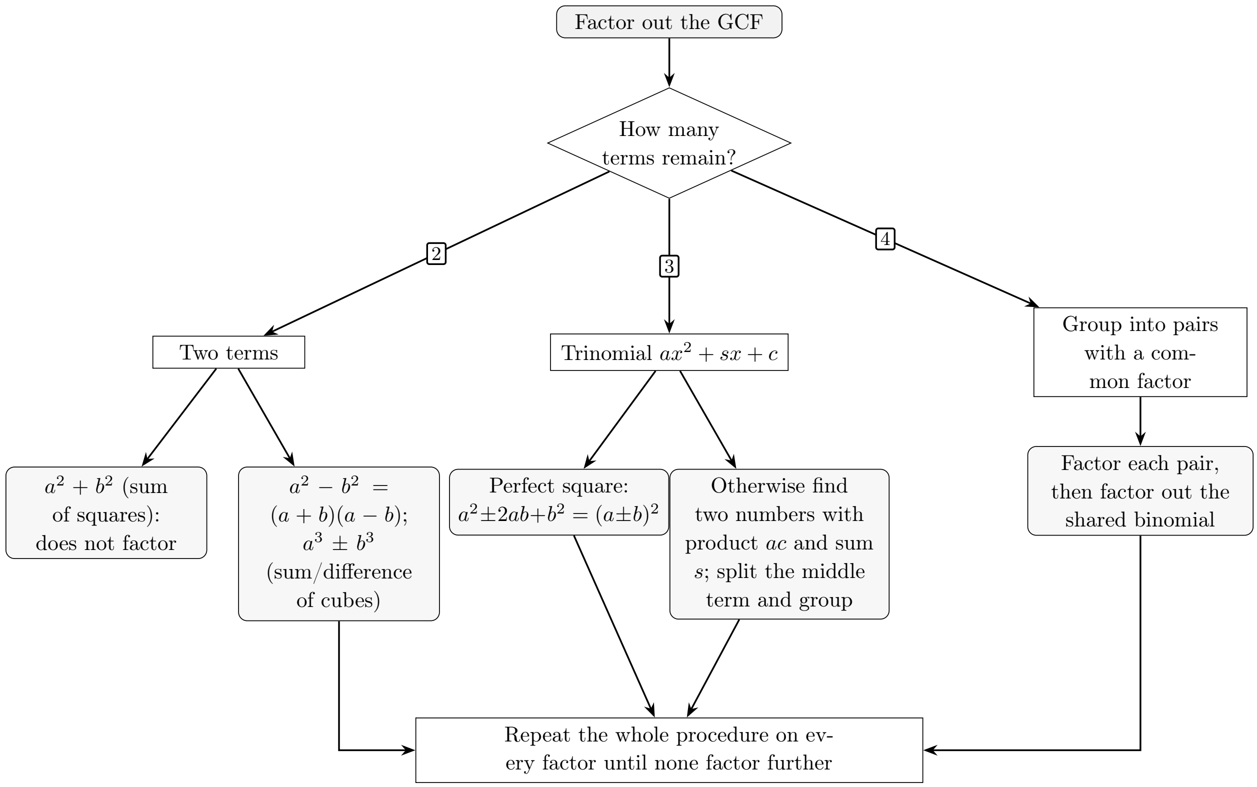
Example 1. Factor $2x^2 + 7x + 3$.

Here $a = 2$, $c = 3$, so $ac = 6$; we need two numbers multiplying to 6 and adding to 7, namely 6 and 1. Split the middle term: $2x^2 + 6x + x + 3 = 2x(x + 3) + 1(x + 3) = (x + 3)(2x + 1)$.

Example 2. Factor completely $3x^3 - 12x$.

First the GCF: $3x^3 - 12x = 3x(x^2 - 4)$. The remaining factor is a difference of squares, so $x^2 - 4 = (x + 2)(x - 2)$. Hence $3x^3 - 12x = 3x(x + 2)(x - 2)$.

A strategy. The flow chart below collects the methods above into a single decision procedure. Always start with the GCF, then branch on how many terms remain, and keep factoring each piece until nothing factors further. If the reader has trouble getting started on factoring, I suggest they use this chart, although the need for it should phase out pretty quickly.



1.6.1 Exercises

1. Factor $5x^2 - 10x$.
2. Factor $12x^3 + 8x^2 - 4x$.
3. Factor $x^2 + 9x + 20$.
4. Factor $x^2 - 3x - 28$.
5. Factor $x^2 - 49$.
6. Factor $9x^2 - 25$.
7. Factor $x^2 + 10x + 25$.
8. Factor $4x^2 - 12x + 9$.
9. Factor $2x^2 + 5x - 3$.
10. Factor $6x^2 - x - 2$.
11. Factor by grouping: $x^3 + 3x^2 + 2x + 6$.
12. Factor by grouping: $2x^3 - x^2 - 6x + 3$.
13. Factor $x^3 - 27$.
14. Factor $8x^3 + 1$.

15. Factor completely $x^4 - 16$.
16. Factor completely $2x^4 - 32$.
17. (☕) Explain why $x^2 + 4$ cannot be factored over the real numbers, but $x^2 - 4$ can.
18. (☕) A student factors $x^2 - 9$ as $(x - 3)(x - 3)$. Identify the error and state the correct factorization, explaining the difference between a difference of squares and a perfect square.
19. (☕) If $x = 5$ is a solution of a polynomial equation $p(x) = 0$, what linear factor must $p(x)$ contain? Connect your answer to the zero-product property.

Chapter 2

Rational Functions and Equations

2.1 Algebra of Rational Functions

A *rational function* is a quotient of two polynomials; in some sense, it is the analogue of an ordinary fraction of integers. Everything you know about arithmetic fractions carries over, with one new feature that must never be forgotten: since we cannot divide by zero, when we view a rational function as a function[1], we have some restrictions on the domain, excluding any real value that makes the denominator zero.

Simplifying. Analogously to the integer case, a fraction is simplified by dividing numerator and denominator by their common factors. The same is true here, with the *caveat* that you may cancel only factors, never individual terms of a sum. For example, $\frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x + 2$.
When a rational function is such that numerator and denominator share no common factors, we say that it is *reduced*.

Operations on rational functions Operations on rational functions work just the same as with rational numbers: if a, b, c and d are polynomials, with $b, d \neq 0$, we can add a/b and c/d by doing

$$\frac{a}{b} + \frac{c}{d} := \frac{ad + bc}{bd},$$

and we can multiply them as

$$\frac{a}{b} \cdot \frac{c}{d} := \frac{ac}{bd}.$$

Division of rational functions is done by remembering that division is multiplication by the inverse,

$$\frac{a}{b} \div \frac{c}{d} := \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}.$$

[1]A rational function is not a function; it is a formal object, like a polynomial. We can construct a function by choosing a rational function as a "rule".

Example 1. Simplify $\dfrac{x^2 - x - 6}{x^2 - 9}$.

Factor top and bottom: $\dfrac{(x-3)(x+2)}{(x-3)(x+3)} = \dfrac{x+2}{x+3}$, valid for $x \neq 3$ and $x \neq -3$.

Example 2. Compute $\dfrac{x^2 - 1}{x+4} \cdot \dfrac{x^2 + 4x}{x^2 + 2x + 1}$.

Factor each piece:

$$\frac{(x-1)(x+1)}{x+4} \cdot \frac{x(x+4)}{(x+1)^2} = \frac{(x-1)(x+1)\,x\,(x+4)}{(x+4)(x+1)^2} = \frac{x(x-1)}{x+1},$$

after cancelling the common factors $x + 4$ and one factor $x + 1$.

2.1.1 Exercises

1. Simplify $\dfrac{6x^2}{9x}$ and state the restriction.
2. Simplify $\dfrac{x^2 + 5x}{x}$.
3. Simplify $\dfrac{x - 3}{x^2 - 9}$.
4. Simplify $\dfrac{x^2 - 25}{x^2 + 10x + 25}$.
5. Simplify $\dfrac{2x^2 - 2}{x^2 + 2x + 1}$.
6. Simplify $\dfrac{x^2 - x - 12}{x^2 - 16}$.
7. Multiply $\dfrac{3}{x} \cdot \dfrac{x^2}{6}$.
8. Multiply $\dfrac{x+1}{x-2} \cdot \dfrac{x-2}{x+3}$.
9. Multiply $\dfrac{x^2 - 4}{x+3} \cdot \dfrac{x^2 + 3x}{x-2}$.
10. Divide $\dfrac{x}{x+1} \div \dfrac{x^2}{x+1}$.
11. Divide $\dfrac{x^2 - 9}{x} \div \dfrac{x-3}{x^2}$.
12. Divide $\dfrac{x^2 + 5x + 6}{x^2 - 4} \div \dfrac{x+3}{x-2}$.
13. Simplify $\dfrac{4 - x}{x - 4}$.
14. Simplify $\dfrac{x^3 - 8}{x - 2}$.

15. Multiply and simplify $\dfrac{x^2-x-6}{x^2-1}\cdot\dfrac{x^2+2x+1}{x^2-9}$.

16. (☕) Explain why $\dfrac{x+5}{5}$ does *not* simplify to x, while $\dfrac{5x}{5}$ does simplify to x.
Hint: If two rational functions are equal, they would have to induce the same function on the reals. Use extensionality to conclude.

2.2 LCMs and LCDs of Rational Functions

While adding fractions requires no particular operations other than addition itself, it is often more efficient to add fractions in a reduced form, by using the least common multiple of the denominators, called the *least common denominator*, LCD for short. The only skill we need to learn is finding a common denominator built from polynomials rather than numbers.

Least common multiple. The least common multiple (LCM) of some polynomials is the lowest-degree polynomial that each one divides. To build it, factor each polynomial completely and take each distinct factor to the highest power in which it appears. The *least common denominator* (LCD) of some rational functions is the LCM of their denominators.

Reduced sum of fractions To do a more efficient sum of fractions, we rewrite each fraction with the LCD by multiplying its numerator and denominator by whatever factors it is missing, then add or subtract the numerators over the common denominator. Be especially careful when subtracting: the minus sign distributes across *every* term of the second numerator. Let us lead by example:

Example 1. Add $\dfrac{2}{x}+\dfrac{3}{x+1}$.

The denominators share no factors, so the LCD is $x(x+1)$:

$$\frac{2}{x}+\frac{3}{x+1}=\frac{2(x+1)}{x(x+1)}+\frac{3x}{x(x+1)}=\frac{2x+2+3x}{x(x+1)}=\frac{5x+2}{x(x+1)}.$$

Example 2. Subtract $\dfrac{x}{x-2}-\dfrac{4}{x^2-4}$.

Factor the second denominator: $x^2-4=(x-2)(x+2)$, so the LCD is $(x-2)(x+2)$:

$$\frac{x(x+2)}{(x-2)(x+2)}-\frac{4}{(x-2)(x+2)}=\frac{x^2+2x-4}{(x-2)(x+2)}.$$

2.2.1 Exercises

1. Find the LCM of x^2 and x^3.

2. Find the LCM of $x(x+1)$ and $(x+1)^2$.

3. Find the LCM of $x^2 - 9$ and $x + 3$.
4. Add $\dfrac{1}{x} + \dfrac{2}{x}$.
5. Add $\dfrac{3}{x} + \dfrac{5}{2x}$.
6. Add $\dfrac{1}{x} + \dfrac{1}{x+2}$.
7. Subtract $\dfrac{5}{x-1} - \dfrac{2}{x}$.
8. Add $\dfrac{2}{x+3} + \dfrac{4}{x-3}$.
9. Subtract $\dfrac{x}{x+1} - \dfrac{1}{x+1}$.
10. Add $\dfrac{x}{x-2} + \dfrac{2}{2-x}$.
11. Subtract $\dfrac{3}{x^2-1} - \dfrac{1}{x-1}$.
12. Add $\dfrac{1}{x^2-x} + \dfrac{1}{x}$.
13. Subtract $\dfrac{x+1}{x^2-4} - \dfrac{1}{x+2}$.
14. Add $\dfrac{2}{x^2+5x+6} + \dfrac{3}{x^2+4x+3}$.
15. Subtract $\dfrac{1}{x} - \dfrac{1}{x+1}$.
16. (☕) Argue that the answer to Exercise 15 induces a function which is always positive for $x > 0$, without plugging in numbers, by inspecting the simplified form.

2.3 Simplifying Nested Rational Functions

A *nested rational function* is a rational function whose numerator or denominator (or both) themselves contain rational functions, a fraction built out of fractions. These appear naturally in many fields, such as combining the resistances of parallel electrical components in physics. The goal is always to rewrite such a tangle as a single, "simpler" rational function. As usual, we have more ways to go about this.

Two reliable methods.

- **Combine, then divide.** Write the numerator as one fraction and the denominator as one fraction, then divide by multiplying by the reciprocal.
- **Multiply by the overall LCD.** Find the LCD of *every* small fraction appearing anywhere in the expression, and multiply the entire top and bottom by that LCD. This clears all the inner fractions in one stroke.

The second method is usually faster, but you have to see the smart choice to make. The first is simply based on the definition of division, and it works without having to make choices.

Example 1 (LCD method). Simplify $\dfrac{\frac{1}{x}+\frac{1}{2}}{\frac{1}{x}-\frac{1}{2}}$.

The inner fractions involve denominators x and 2, so the overall LCD is $2x$. Multiply top and bottom by $2x$:

$$\frac{\left(\frac{1}{x}+\frac{1}{2}\right)2x}{\left(\frac{1}{x}-\frac{1}{2}\right)2x} = \frac{2+x}{2-x}.$$

Example 2 (Combine then divide). Simplify $\dfrac{1-\frac{1}{x}}{x-\frac{1}{x}}$.

Write each part over x: numerator $= \dfrac{x-1}{x}$, denominator $= \dfrac{x^2-1}{x}$. Then

$$\frac{x-1}{x} \div \frac{x^2-1}{x} = \frac{x-1}{x} \cdot \frac{x}{x^2-1} = \frac{x-1}{(x-1)(x+1)} = \frac{1}{x+1}.$$

2.3.1 Exercises

1. Simplify $\dfrac{\frac{2}{3}}{\frac{4}{9}}$.
2. Simplify $\dfrac{\frac{1}{x}}{\frac{1}{x^2}}$.
3. Simplify $\dfrac{\frac{x}{2}}{\frac{x}{6}}$.
4. Simplify $\dfrac{1+\frac{1}{x}}{1-\frac{1}{x}}$.
5. Simplify $\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x}-\frac{1}{y}}$.
6. Simplify $\dfrac{x+\frac{1}{x}}{x-\frac{1}{x}}$.
7. Simplify $\dfrac{\frac{1}{x}-\frac{1}{3}}{x-3}$.
8. Simplify $\dfrac{\frac{1}{x+1}}{\frac{1}{x^2-1}}$.
9. Simplify $\dfrac{2-\frac{3}{x}}{4-\frac{9}{x^2}}$.

10. Simplify $\dfrac{\frac{1}{x}+\frac{1}{x+1}}{\frac{1}{x}-\frac{1}{x+1}}$.

11. Simplify $\dfrac{\frac{x}{y}-\frac{y}{x}}{\frac{1}{y}-\frac{1}{x}}$.

12. Simplify $\dfrac{\frac{a}{b}+1}{\frac{a}{b}-1}$.

13. Simplify $\dfrac{\frac{1}{x^2}-\frac{1}{9}}{\frac{1}{x}+\frac{1}{3}}$.

14. Two resistors of R_1 and $R_2\,\Omega$ in parallel have combined resistance $\dfrac{1}{\frac{1}{R_1}+\frac{1}{R_2}}$. Simplify this to a single fraction.

15. Simplify $\dfrac{\frac{1}{x+h}-\frac{1}{x}}{h}$.

16. (☕) Solve Exercise 4 by both methods and confirm the answers agree, then state which method you found faster and why.

17. (☕) Why is multiplying top and bottom by the overall LCD a legitimate operation that does not change the value of the expression?

2.4 Solving Rational Equations

We now move from rewriting rational expressions to solving equations that contain them. Work-rate problems, such as "one pipe fills a tank in 3 hours, another in 5; how long together?", lead directly to an equation with variables in the denominators. The strategy is to clean up and solve the resulting equation, but doing so introduces spurious solutions, which we must exclude. First, let us recall what an equality of two fractions actually means:

Equality of fractions. Recall that, *by definition*, given two fractions $\frac{a}{b}$ and $\frac{c}{d}$, saying that

$$\frac{a}{b}=\frac{c}{d}$$

means that $ad = bc$. For example,

$$\frac{2}{3}=\frac{4}{6},$$

because $2 \cdot 6 = 4 \cdot 3 = 12$. The same holds for rational functions, and fractions of every kind.

Conditions on the domain. A rational function, *per se*, has no domain, simply being a formal object. However, when we solve an equation involving a rational function, we have to make sure that evaluating the equation on numbers makes no denominators vanish. For example, evaluating the equation

$$\frac{x-4}{x}=0$$

on $x = 4$, $x = 2$ and $x = 0$ respectively yields

$$\frac{0}{4} = 0, \quad \frac{-2}{2} = 0, \quad \frac{-4}{0} = 0$$

respectively. The first two statements make sense, even if the second one is false, but the third is not well-defined, since denominators of fractions cannot be zero *by definition.* Thus, when solving a rational equation, we have to make sure to exclude any such values, imposing some conditions on the domain. In the previous example, we *must* ask that $x \neq 4$.

Strategy. The strategy to solve such an equation is simple: carry out all operations on the rational functions involved, so as to end up with an equation of the form

$$\frac{p(x)}{q(x)} = 0;$$

then, applying the definition of equality of fractions will give us a polynomial equation $p(x) = 0$ which we can solve using the method of choice. To see this, we can rewrite 0 as $0/1$, and applying the definition above, we get

$$\frac{p(x)}{q(x)} = \frac{0}{1},$$

which means that

$$p(x) = p(x) \cdot 1 = q(x) \cdot 0 = 0,$$

i.e.,

$$p(x) = 0.$$

Thus, we can say with a catchphrase that "any rational function $p(x)/q(x)$ is zero if and only if its numerator is".

Example 1. Solve $\dfrac{1}{x} + \dfrac{1}{2} = \dfrac{3}{x}$.

Following the strategy, we move everything to one side and carry out the operations, so as to end up with a single fraction equal to 0:

$$\frac{1}{x} + \frac{1}{2} - \frac{3}{x} = \frac{2}{2x} + \frac{x}{2x} - \frac{6}{2x} = \frac{x-4}{2x} = 0.$$

Now we apply the definition of equality of fractions: a rational function is zero if and only if its numerator is, so

$$x - 4 = 0, \qquad \text{i.e.} \qquad x = 4.$$

Since $x = 4$ makes no denominator zero, it is a valid solution.

Example 2. Solve $\dfrac{x}{x-3} = \dfrac{3}{x-3} + 2$.

Again we collect everything on one side and combine into a single fraction:

$$\frac{x}{x-3} - \frac{3}{x-3} - 2 = \frac{x-3-2(x-3)}{x-3} = \frac{x-3-2x+6}{x-3} = \frac{3-x}{x-3} = 0.$$

Applying the definition of equality of fractions, the numerator must vanish:

$$3 - x = 0, \qquad \text{i.e.} \qquad x = 3.$$

But $x = 3$ makes the denominator $x-3$ zero, so this candidate is *extraneous.* The equation therefore has *no solution.*

2.4.1 Exercises

1. Solve $\dfrac{x}{4} = \dfrac{3}{2}$.
2. Solve $\dfrac{2}{x} = \dfrac{1}{3}$.
3. Solve $\dfrac{1}{x} + 1 = \dfrac{3}{x}$.
4. Solve $\dfrac{5}{x} - \dfrac{2}{x} = 6$.
5. Solve $\dfrac{x+1}{x} = 2$.
6. Solve $\dfrac{3}{x-1} = \dfrac{6}{x+2}$.
7. Solve $\dfrac{1}{x} + \dfrac{1}{x+1} = \dfrac{1}{2}$.
8. Solve $\dfrac{x}{x-2} + 1 = \dfrac{2}{x-2}$.
9. Solve $\dfrac{4}{x-3} = \dfrac{x}{x-3} + 2$.
10. Solve $\dfrac{1}{x-1} + \dfrac{1}{x+1} = \dfrac{2}{x^2-1}$.
11. Solve $\dfrac{x}{x+2} - \dfrac{2}{x-2} = \dfrac{8}{x^2-4}$.
12. Solve $\dfrac{6}{x} + \dfrac{6}{x-5} = 5$.
13. Solve $\dfrac{2x}{x+1} = 2 - \dfrac{2}{x+1}$.
14. One pipe fills a pool in 6 h, another in 4 h. Set up and solve an equation for the time t to fill it together.
15. A boat goes 12 mi upstream in the same time it goes 18 mi downstream; the current is 2 mph. Set up a rational equation for the boat's still-water speed and solve it.

Chapter 3

Radical Expressions, Exponents, and Equations

3.1 Radical Expressions and Functions

Roots arise naturally in geometry: given a square, we want a way to find its side given the area; *taking a root* is the operation that inverts raising to a power, and it brings with it a new family of interesting functions.

Definition (n-th root of a number). For a positive integer n, an *n-th root* of a number a is a number b such that $b^n = a$.

A Motivating Example. Let $n = 2$. In this case, the 2-nd root of a number is called the *square root*. Now, let us consider any number of our liking, let's say 4. It is clear that 4 has two square roots, namely 2 and -2, and the same goes for any positive number that admits one: in fact, if a admits b as square root,

$$(-b)^2 = (-1) \cdot (-1) \cdot b^2 = 1 \cdot a = a,$$

which means that $-b$ is another square root for a.
To say this with our newly found language, "taking a real number and sending it into its square root" does *not* define a function: there is a choice to make.

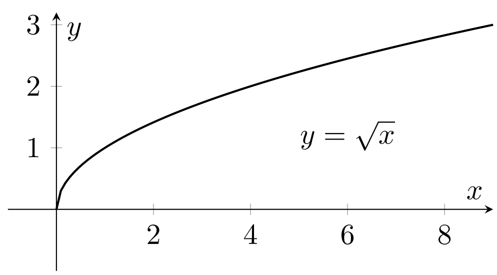
Principal Roots. We thus make a choice, and we make it coherently for all real numbers: given a positive real number a, we define its *principal* square root to be the positive one, and we denote it by $\sqrt{a}$. We have now defined a function f, which we will call the *square root* function, that takes positive reals as input, and outputs positive reals again:

$$f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, \quad f(x) = \sqrt{x}.$$

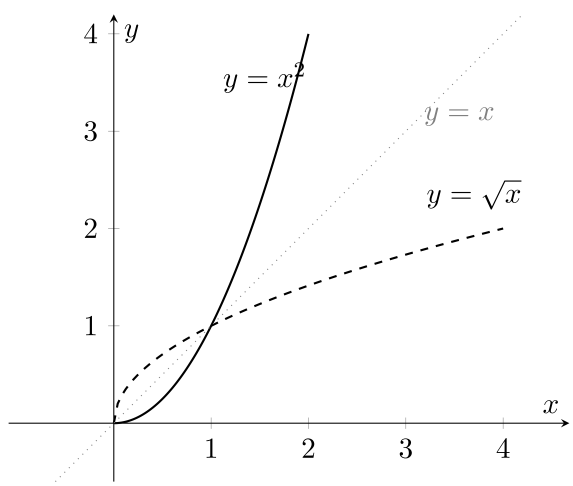
Odd Roots. On the contrary, odd-root functions are always defined, even for negative numbers. For example, $\sqrt[3]{-8} = -2$.

A remark. Because the root function always returns the principal root of the argument, $\sqrt{x^2} = |x|$, not simply x. For example $\sqrt{(-5)^2} = \sqrt{25} = 5 = |-5|$. This absolute value is easy to overlook and is a frequent source of error.

Radical functions and their domains. The function $f(x) = \sqrt[n]{x}$ is a *radical function.* When n is even, the argument must be nonnegative, so the domain cannot include inputs that make the inside negative; when n is odd there is no such restriction. The graph of $f(x) = \sqrt{x}$ below begins at the origin and rises ever more slowly, it is the upper half of a sideways parabola.



In fact, the graph of the inverse function of a given function f is the reflection of the graph of f through the line $y = x$.



Example 1. Find the domain of $f(x) = \sqrt{2x - 6}$, and express it in interval notation.

The radicand must be nonnegative: $2x - 6 \geq 0$, so $x \geq 3$. The domain is $[3, \infty)$.

3.1.1 Exercises

1. Evaluate $\sqrt{64}$.

2. Evaluate $\sqrt[3]{125}$.
3. Evaluate $\sqrt[3]{-27}$.
4. Evaluate $\sqrt[4]{16}$.
5. Is $\sqrt{-9}$ a real number? Explain.
6. Simplify $\sqrt{(-7)^2}$.
7. Simplify $\sqrt{x^2}$ for a general real x.
8. Simplify $\sqrt[3]{x^3}$ for a general real x.
9. Find the domain of $f(x) = \sqrt{x-5}$.
10. Find the domain of $f(x) = \sqrt{7-x}$.
11. Find the domain of $f(x) = \sqrt{3x+9}$.
12. Find the domain of $f(x) = \sqrt[3]{x-1}$.
13. Find the domain of $f(x) = \sqrt{x^2-4}$.
14. Evaluate $f(x) = \sqrt{x+1}$ at $x = 0, 3, 8$.
15. Find the domain and range of $f(x) = \sqrt{x} + 2$.
16. (☕) Compare the domains of $\sqrt[3]{x}$ and $\sqrt{x}$ and explain the difference in terms of which inputs produce real outputs.
17. (☕) Why does the graph of $y = \sqrt{x}$ contain no points below the x-axis, even though 4 has -2 as a square root?

3.2 Rational Numbers as Exponents

So far exponents have been integer numbers, recording repeated multiplication. But there is a compelling reason to allow fractional exponents: doing so lets the familiar laws of exponents, which we want to keep, tell us what a symbol like $a^{1/2}$ must mean. This unification turns every root into an exponent, so that radicals and powers become one notation.

Defining $a^{1/n}$. We want the law $(a^m)^n = a^{mn}$ to hold for fractional exponents, then $\left(a^{1/n}\right)^n = a^{(1/n)\cdot n} = a^1 = a$ has to hold. So $a^{1/n}$ must be a number whose nth power is a, that is, an nth root. We therefore *define*

$$a^{1/n} = \sqrt[n]{a}, \qquad \text{and more generally} \qquad a^{m/n} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}.$$

This is the only definition consistent with the existing exponent rules, which is exactly why we adopt it. Moreover, we will see how these requests are (almost) enough to define a particular family of functions, called *exponentials*.

All the laws still hold. With this definition, the usual rules, $a^r a^s = a^{r+s}$, $\dfrac{a^r}{a^s} = a^{r-s}$, $(a^r)^s = a^{rs}$, $a^{-r} = \dfrac{1}{a^r}$, remain valid for all rational exponents. Therefore, all radical expressions and equation will have a handier exponential form, which you are encouraged to learn how to manipulate.

Example 1. Evaluate $27^{2/3}$.

Take the cube root first, then square: $27^{2/3} = \left(\sqrt[3]{27}\right)^2 = 3^2 = 9$. (Rooting before powering keeps the intermediate steps more manageable, since the numbers will be smaller.)

Example 2. Simplify $\dfrac{x^{1/2}\,x^{1/3}}{x^{1/6}}$ using exponent laws.

$$\frac{x^{1/2}x^{1/3}}{x^{1/6}} = x^{1/2+1/3-1/6} = x^{3/6+2/6-1/6} = x^{4/6} = x^{2/3}.$$

3.2.1 Exercises

1. Write $\sqrt{x}$ using a rational exponent.
2. Write $\sqrt[3]{x^2}$ using a rational exponent.
3. Write $x^{3/4}$ in radical form.
4. Evaluate $16^{1/2}$.
5. Evaluate $8^{1/3}$.
6. Evaluate $16^{3/4}$.
7. Evaluate $32^{2/5}$.
8. Evaluate $25^{-1/2}$.
9. Evaluate $\left(\frac{8}{27}\right)^{2/3}$.
10. Simplify $x^{1/2} \cdot x^{1/2}$.
11. Simplify $\dfrac{x^{3/4}}{x^{1/4}}$.
12. Simplify $\left(x^{2/3}\right)^6$.
13. Simplify $\left(8x^6\right)^{1/3}$.
14. Simplify $\dfrac{x^{1/2}y^{2/3}}{x^{-1/2}y^{1/3}}$.
15. Simplify $\sqrt[6]{x^3}$ by converting to a rational exponent.
16. (☕) Explain why the definition $a^{1/n} = \sqrt[n]{a}$ is forced upon us if we insist that $(a^r)^s = a^{rs}$ continue to hold.

17. (☕) For evaluating $64^{2/3}$, explain why taking the root before the power is easier than the power before the root, even though both give the same result.
18. (☕) Why is $(-8)^{1/3}$ a real number while $(-8)^{1/2}$ is not? Relate this to the parity of the denominator of the exponent.

3.3 Simplifying Radical Expressions

A radical expression, like a fraction, has a preferred "simplest" form, and putting it there makes expressions easier to compare, combine, and recognize as equal. The tools are two product-and-quotient rules that follow directly from the exponent laws of the previous section.

The product and quotient rules. For nonnegative radicands (and any radicand when the index is odd),

$$\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}, \qquad \sqrt[n]{\tfrac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

These are simply $(ab)^{1/n} = a^{1/n}b^{1/n}$ rewritten in radical notation.

Simplified form. A radical is in simplified form when no factor of the radicand is a perfect power, no fractions remain under the radical; some people ask that no radicals remain in any denominator, but we will not. The basic technique is to find the largest perfect power and just factor it out: $\sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{25}\sqrt{2} = 5\sqrt{2}$.

Example 1. Simplify $\sqrt{72}$.

Factor out the largest perfect square: $72 = 36 \cdot 2$, so $\sqrt{72} = \sqrt{36}\sqrt{2} = 6\sqrt{2}$.

Example 2. Simplify $\sqrt{18x^3y^2}$, assuming $x, y \geq 0$.

Group perfect squares: $18x^3y^2 = (9x^2y^2)(2x)$, so

$$\sqrt{18x^3y^2} = \sqrt{9x^2y^2} \cdot \sqrt{2x} = 3xy\sqrt{2x}.$$

3.3.1 Exercises

1. Simplify $\sqrt{12}$.
2. Simplify $\sqrt{45}$.
3. Simplify $\sqrt{75}$.
4. Simplify $\sqrt{200}$.
5. Simplify $\sqrt[3]{54}$.

6. Simplify $\sqrt[3]{40}$.
7. Simplify $\sqrt{x^5}$ (assume $x \geq 0$).
8. Simplify $\sqrt{50x^2}$ (assume $x \geq 0$).
9. Simplify $\sqrt{48x^3}$ (assume $x \geq 0$).
10. Simplify $\sqrt[3]{16x^4}$.
11. Simplify $\sqrt{\dfrac{9}{16}}$.
12. Simplify $\sqrt{\dfrac{x^2}{25}}$ (assume $x \geq 0$).
13. Simplify $\sqrt{32x^4y^5}$ (assume $x, y \geq 0$).
14. Simplify $\sqrt[3]{27x^6y^3}$.
15. Simplify $\sqrt{98a^3b^4}$ (assume $a, b \geq 0$).
16. (☕) Explain why $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$ in general, and give a numerical example that exposes the error.
17. (☕) Without assuming $x \geq 0$, simplify $\sqrt{x^2}$ correctly, and explain why the absolute value is necessary here but not in $\sqrt{x^4}$.
18. (☕) Explain how the product rule for radicals is just the exponent law $(ab)^{1/n} = a^{1/n}b^{1/n}$ in disguise.

3.4 Addition, Subtraction, and Multiplication of Radical Expressions

Radicals combine under addition and multiplication according to rules that should feel familiar: addition works like combining like terms, and multiplication is done with the distributive law, with the product rule handling the radicands. For all intents and purposes, a term like $\sqrt{a}$ will behave much like a variable in a polynomial.

Adding and subtracting like radicals. Two radical terms are *like* if they have the same index and the same radicand. Only like radicals may be combined, and they combine by adding coefficients, exactly as $3x + 5x = 8x$: thus $3\sqrt{2} + 5\sqrt{2} = 8\sqrt{2}$. Crucially, you often have to *simplify first* to reveal like radicals: for example, $\sqrt{8} + \sqrt{2} = 2\sqrt{2} + \sqrt{2} = 3\sqrt{2}$.

Multiplying. Use the distributive law, then the product rule to merge radicands and simplify. A particularly useful product is that of *conjugates*:

$$(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = (\sqrt{a})^2 - (\sqrt{b})^2 = a - b,$$

which is a difference of squares that eliminates the radicals entirely.

Example 1. Simplify $\sqrt{27} + 2\sqrt{12}$.

Simplify each radical first: $\sqrt{27} = 3\sqrt{3}$ and $2\sqrt{12} = 2 \cdot 2\sqrt{3} = 4\sqrt{3}$. Now they are like, and so we can write $3\sqrt{3} + 4\sqrt{3} = 7\sqrt{3}$.

Example 2. Multiply $(\sqrt{5} + 2)(\sqrt{5} - 3)$.

$\sqrt{5} \cdot \sqrt{5} - 3\sqrt{5} + 2\sqrt{5} - 6 = 5 - \sqrt{5} - 6 = -1 - \sqrt{5}$.

3.4.1 Exercises

1. Combine $2\sqrt{3} + 5\sqrt{3}$.
2. Combine $7\sqrt{x} - 3\sqrt{x}$.
3. Combine $\sqrt{8} + \sqrt{18}$.
4. Combine $\sqrt{50} - \sqrt{32}$.
5. Combine $\sqrt{12} + \sqrt{27} - \sqrt{3}$.
6. Combine $2\sqrt{20} + 3\sqrt{45}$.
7. Combine $\sqrt[3]{16} + \sqrt[3]{54}$.
8. Multiply $\sqrt{3} \cdot \sqrt{6}$.
9. Multiply $\sqrt{2}\,(\sqrt{8} + \sqrt{2})$.
10. Multiply $(\sqrt{3} + 1)(\sqrt{3} - 1)$.
11. Multiply $(\sqrt{5} + \sqrt{2})(\sqrt{5} - \sqrt{2})$.
12. Multiply $(\sqrt{x} + 3)^2$ (assume $x \geq 0$).
13. Multiply $(2\sqrt{3} - 1)(\sqrt{3} + 4)$.
14. Multiply $(\sqrt{7} + \sqrt{3})^2$.
15. Combine $\sqrt{45x} - \sqrt{20x}$ (assume $x \geq 0$).
16. (☕) Explain why $\sqrt{2} + \sqrt{3}$ cannot be combined into a single radical term, while $\sqrt{8} + \sqrt{2}$ can.
17. (☕) Show that the product of conjugates $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$ contains no radicals, and explain why this is the key to rationalizing.
18. (☕) A student writes $(\sqrt{x} + 3)^2 = x + 9$. Identify the missing term and explain the error using the perfect-square pattern.

3.5 Solving Radical Equations

A *radical equation* contains the variable under a radical, as in $\sqrt{x+1} = 3$. Such equations appear whenever a model is stated in terms of a root: the period of a pendulum, for instance, is proportional to the square root of its length, so recovering the length from a measured period requires solving a radical equation. The natural move is to undo the root by raising both sides to a power, but this step can introduce spurious solutions.

Strategy. Isolate a radical on one side, then raise both sides to the power equal to the index, eliminating that radical. Repeat if more than one radical is present. Solve the resulting polynomial equation, and, this is essential, check every candidate in the *original* equation.

An example. Solve $\sqrt{x+2}+4=x$.

First isolate the radical by subtracting 4 from both sides:

$$\sqrt{x+2}=x-4.$$

Now square both sides to eliminate the root:

$$x+2=(x-4)^2=x^2-8x+16,$$

which rearranges to the equation

$$0=x^2-9x+14=(x-2)(x-7),$$

so the candidates are $x=2$ and $x=7$. Finally, check each in the *original* equation. For $x=7$: $\sqrt{7+2}+4=\sqrt{9}+4=3+4=7$, and so 7 is a solution. For $x=2$: $\sqrt{2+2}+4=\sqrt{4}+4=2+4=6\neq 2$, so $x=2$ is not a solution, and must be discarded. The only solution is $x=7$.

Why extraneous solutions appear. Squaring is not a reversible operation: distinct numbers can have equal squares, since $a=b$ implies $a^2=b^2$ but not conversely (for example $-3\neq 3$ yet $(-3)^2=3^2$). So squaring both sides can produce an equation with *more* solutions than the original, and the extra ones must be discarded by checking.

Example 1. Solve $\sqrt{x+1}=3$.

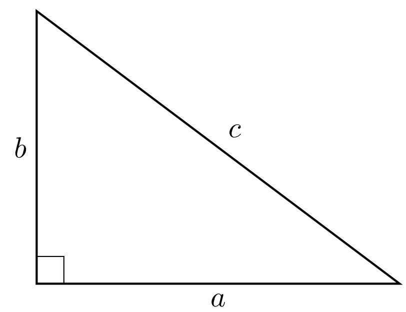
Square both sides: $x+1=9$, so $x=8$. Check: $\sqrt{8+1}=\sqrt{9}=3$. ✓Valid.

Example 2. Solve $\sqrt{x+7}=x+1$.

Square: $x+7=(x+1)^2=x^2+2x+1$, so $0=x^2+x-6=(x+3)(x-2)$, giving $x=-3$ or $x=2$. Check each in the original: for $x=2$, $\sqrt{9}=3$ and $x+1=3$ ✓; for $x=-3$, $\sqrt{4}=2$ but $x+1=-2$, and $2\neq -2$, so $x=-3$ is *extraneous*. The only solution is $x=2$.

The Pythagorean Theorem. One of the most natural instances of square roots in geometry is in the Pythagorean Theorem. In a right triangle[1], call the two shorter sides, the ones forming the right angle, the *legs*, and call a and b their lengths; call the longest side, the one opposite the right angle, the *hypotenuse*, and call c its length.

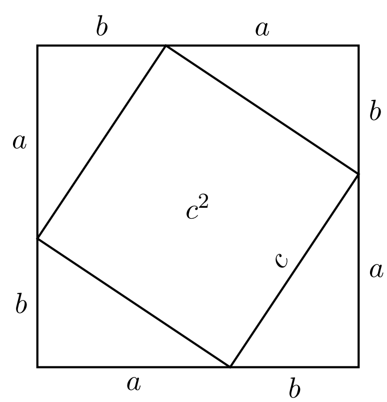
[1] A right triangle is a triangle which has a 90° angle.



We prove that the lengths of the three sides are related by

$$a^2 + b^2 = c^2.$$

Proof. The proof is a simple geometric construction: take four copies of the right triangle above, with legs a, b and hypotenuse c, and arrange them inside a square of side $a+b$, each rotated a quarter turn from the next, so that their hypotenuses trace out a smaller, tilted square in the middle, as in the picture below.



The big square has area $(a+b)^2$. This same area is also the sum of the four congruent triangles, each of area $\frac{1}{2}ab$, together with the inner tilted square, of area c^2 (it is a square because each of its four sides has length c, and one can check its interior angles are right angles, since the two acute angles of the triangle meeting at each vertex add up to 90°). So

$$(a+b)^2 = 4\left(\tfrac{1}{2}ab\right) + c^2 = 2ab + c^2.$$

Expanding the left side, $a^2 + 2ab + b^2 = 2ab + c^2$, and cancelling $2ab$ from both sides leaves

$$a^2 + b^2 = c^2,$$

as claimed.

Example 3. A right triangle has legs $a = 3$ and $b = 4$. Find the hypotenuse c.

By the theorem, $c = \sqrt{a^2 + b^2} = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$.

Example 4. A right triangle has legs $a = 6$ and $b = 12$. Find the hypotenuse c.

Again $c = \sqrt{a^2 + b^2} = \sqrt{36 + 144} = \sqrt{180}$, and since this radical is not yet simplified, we factor out the largest perfect square, $180 = 36 \cdot 5$, to get $c = \sqrt{36 \cdot 5} = 6\sqrt{5}$.

3.5.1 Exercises

1. Solve $\sqrt{x} = 5$.
2. Solve $\sqrt{x} = -2$.
3. Solve $\sqrt{x - 3} = 4$.
4. Solve $\sqrt{2x + 1} = 3$.
5. Solve $\sqrt[3]{x} = 2$.
6. Solve $\sqrt[3]{x - 1} = -2$.
7. Solve $\sqrt{x} + 3 = 7$.
8. Solve $\sqrt{3x - 2} = x$.
9. Solve $\sqrt{x + 6} = x$.
10. Solve $\sqrt{x + 5} = x - 1$.
11. Solve $\sqrt{2x + 3} = x$.
12. Solve $\sqrt{x} + \sqrt{x + 3} = 3$.
13. Solve $\sqrt{x + 1} = \sqrt{2x - 3}$.
14. Solve $x - \sqrt{x} = 6$ (hint: let $u = \sqrt{x}$).
15. The period (in seconds) of a pendulum of length L feet is $T = 2\pi\sqrt{L/32}$. Find L if $T = 2\pi$.
16. A right triangle has legs $a = 5$ and $b = 12$. Find the hypotenuse c.
17. A right triangle has legs $a = 8$ and $b = 8$. Find the hypotenuse c, simplifying the radical.
18. A right triangle has hypotenuse $c = 13$ and one leg $a = 5$. Find the other leg b.
19. (☕) Explain how you could tell, before doing any algebra, that $\sqrt{x} = -2$ has no solution. *Hint:* Think of a square with area x.
20. (☕) Why is checking solutions an unavoidable step for radical equations with even-index radicals, while it is not for odd-index radicals?

Chapter 4

Second-Degree Equations and Polynomial Functions

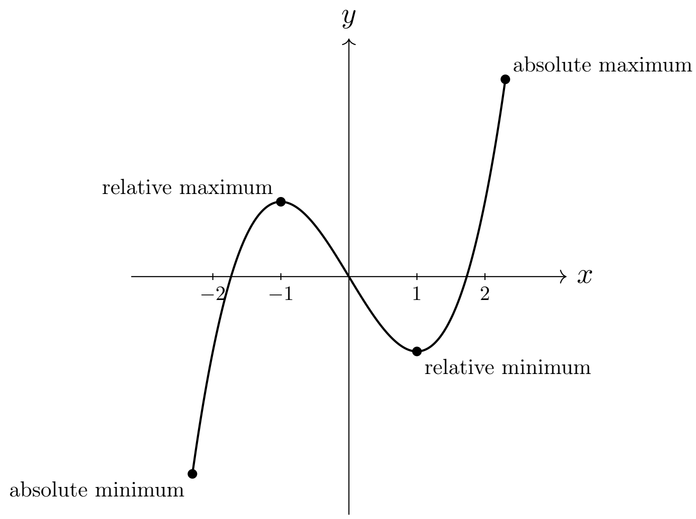
4.1 Increasing, Decreasing, and Piecewise-Defined Functions

Beyond computing values, we want to describe how a function *behaves*: where its graph rises, where it falls, where it turns around. Some functions are even assembled from different formulas on different parts of the domain, and we call these *piecewise defined* function.

Increasing and decreasing functions. We say that a function f is *increasing* on an interval if its outputs grow as the inputs grow: whenever $x_1 < x_2$ in the interval, we have $f(x_1) < f(x_2)$. It is *decreasing* if the inputs growing makes the outputs decrease. Observe that these are statements about *intervals*, not single points, since rising and falling are inherently comparisons between two different inputs and outputs.

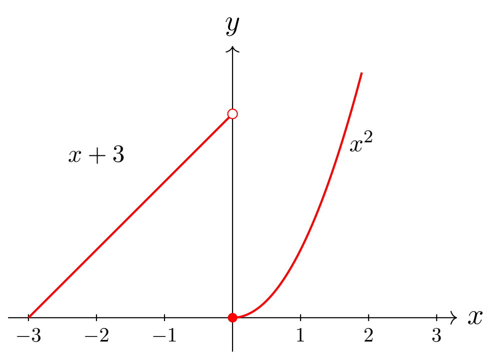
Relative maxima and minima. A point where the function changes from increasing to decreasing is a *relative* (or local) *maximum*; from decreasing to increasing, a *relative minimum*. “Relative” signals that the value is largest or smallest only in comparison with nearby points, not necessarily over the whole domain.

Absolute maxima and minima. By contrast, an *absolute* (or global) *maximum* is the largest value the function attains anywhere on its domain, and an *absolute minimum* is the smallest. A relative extremum need not be an absolute one, and an absolute extremum can occur at an endpoint of the domain, where the function does not turn around at all.



The function graphed above is defined only on a closed interval. It turns from increasing to decreasing at $x = -1$, giving a relative maximum, and from decreasing to increasing at $x = 1$, giving a relative minimum. Neither is absolute: the function climbs higher at the right endpoint and drops lower at the left endpoint, and these two endpoints carry the absolute maximum and the absolute minimum.

Piecewise-defined functions. A piecewise function is given by different formulas on different parts of its domain. To evaluate it at an input, first determine which piece's condition the input satisfies, then use that piece's formula. To graph it, graph each piece only over the inputs where it applies, using a filled dot to include an endpoint and an open dot to exclude one.



The graph above shows $f(x) = x + 3$ for $x < 0$ (open dot at $(0, 3)$, excluded) and $f(x) = x^2$ for $x \geq 0$ (filled dot at $(0, 0)$, included).

Example 1. For the piecewise function $f(x) = \begin{cases} x+3, & x < 0 \\ x^2, & x \geq 0 \end{cases}$, compute $f(-2)$ and $f(3)$.

Since $-2 < 0$, use the first piece: $f(-2) = -2 + 3 = 1$. Since $3 \geq 0$, use the second: $f(3) = 3^2 = 9$.

Example 2. Describe the behavior of $g(x) = x^2$. For $x < 0$, larger inputs give smaller outputs, so g is decreasing on $(-\infty, 0)$; for $x > 0$ it is increasing on $(0, \infty)$. The change from decreasing to increasing at $x = 0$ marks a relative (here, absolute) minimum at $(0, 0)$.

4.1.1 Exercises

1. For $f(x) = \begin{cases} 2x, & x < 1 \\ x + 1, & x \geq 1 \end{cases}$, find $f(0)$, $f(1)$, $f(4)$.
2. For the same f, is the point $x = 1$ taken from the first or second piece? Explain.
3. For $g(x) = \begin{cases} -x, & x \leq 0 \\ x^2, & x > 0 \end{cases}$, find $g(-3)$ and $g(2)$.
4. State the intervals where $f(x) = x^2 - 4$ is increasing and decreasing.
5. State the intervals where $f(x) = -x^2 + 1$ is increasing and decreasing.
6. Where is $f(x) = |x|$ increasing? Decreasing?
7. Identify any relative maximum or minimum of $f(x) = x^2 - 6x + 5$.
8. Sketch $f(x) = \begin{cases} 1, & x < 0 \\ x, & 0 \leq x \leq 2 \\ 2, & x > 2 \end{cases}$ and state where it is constant.
9. For the absolute value function $f(x) = |x - 2|$, find the location of its minimum.
10. Graph $f(x) = \begin{cases} x + 1, & x < 2 \\ 5 - x, & x \geq 2 \end{cases}$ and find $f(2)$.
11. A taxi charges \$4 for the first mile and \$2 per mile thereafter. Write the fare as a piecewise function of miles $m \geq 0$.
12. For $f(x) = x^3$, is the function ever decreasing? Justify.
13. Find the relative minimum value of $f(x) = x^2 + 2x + 5$ by completing the square.
14. Determine whether $f(x) = \begin{cases} x^2, & x \leq 1 \\ 2x - 1, & x > 1 \end{cases}$ has a break (jump) at $x = 1$ by comparing the two pieces there.
15. For the same f as Exercise 14, state the intervals of increase and decrease.
16. (☕) Explain why "increasing" must be defined on an interval rather than at a single point. What would it even mean for a function to be increasing "at" one number?
17. (☕) A function is increasing on $(-\infty, 0)$ and increasing on $(0, \infty)$ but has a jump at 0. Can we conclude it is increasing on all of $(-\infty, \infty)$? Give an example settling the question.
18. (☕) In a piecewise function, why does it matter whether each boundary inequality is strict ($<$) or not ($\leq$)? Describe what goes wrong if both pieces claim the same boundary point.

4.2 Quadratic Equations

When a quantity depends on the square of another, the dependence can be expressed through a second-degree polynomial function, or *quadratic* function. The area of a square depends on the square of its side; the height of a projectile under gravity is a quadratic function of time, and the kinetic energy of a moving body depends on the square of its speed. Solving the equation $ax^2 + bx + c = 0$ becomes thus an important skill to possess and develop.

Zeros and x-intercepts. A *zero* of a function f is an input x with $f(x) = 0$. For $f(x) = ax^2 + bx + c$, the zeros are exactly the x-coordinates where the graph crosses the x-axis. Since the equation that determines the x-axis is $y = 0$, solving the quadratic equation and finding the x-intercepts of the parabola are the same task.

Three solution methods.

- **Factoring:** if $ax^2 + bx + c$ factors as the product of two nonconstant polynomials, set each factor to zero and solve the two resulting first-degree equations.
- **Completing the square:** Rewrite the polynomial equation $ax^2 + bx + c$ as $(x - h)^2 = k$, and take the square roots. This method is general.
- **The quadratic formula:** for any $ax^2 + bx + c = 0$ with $a \neq 0$, the solutions are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

  Since this method is equivalent to completing the square, it is also general.

The discriminant. The quantity $b^2 - 4ac$ under the radical, called the *discriminant*, signals the nature of the solutions without computing them: if it is positive there are two distinct real solutions, if zero there is exactly one repeated root, and if negative there are no real solutions (equivalently, the parabola given by the equation misses the x-axis). It is interesting to point out that the quadratic formula can be thus used as a proof of the fact that a polynomial equation admits no solutions.

False squares are irreducible. We encountered false squares, i.e., trinomials of the form $a^2 \pm ab + b^2$, when factoring sums and differences of cubes. We can now show that false squares are irreducible; it is equivalent to prove that they have no real zeros. Assume $b \neq 0$, since for $b = 0$ the trinomial collapses to a^2, and read $a^2 \pm ab + b^2$ as a quadratic in the variable a, with coefficients 1, $\pm b$ and b^2. If it factored as a product of two nonconstant polynomials, each factor would have first degree in a, and setting each of them to zero would exhibit a real zero of the trinomial. Its discriminant, however, is

$$(\pm b)^2 - 4(1)(b^2) = b^2 - 4b^2 = -3b^2,$$

where the $\pm$ disappears under the square, and $-3b^2 < 0$ for every $b \neq 0$. Thus, there are no real zeros, hence no first-degree factors, and the false square is irreducible.

Example 1 (Factoring). Solve $x^2 - x - 12 = 0$.

Factor: $(x-4)(x+3) = 0$, so $x = 4$ or $x = -3$ by the zero-product property.

Example 2 (Quadratic formula). Solve $2x^2 - 4x - 1 = 0$.

Here $a = 2$, $b = -4$, $c = -1$, so the discriminant is $(-4)^2 - 4(2)(-1) = 16 + 8 = 24 > 0$ (two real roots):

$$x = \frac{4 \pm \sqrt{24}}{4} = \frac{4 \pm 2\sqrt{6}}{4} = \frac{2 \pm \sqrt{6}}{2}.$$

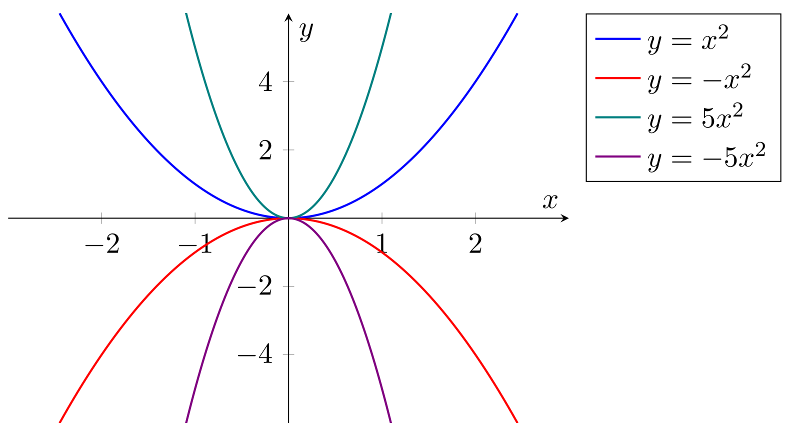
4.2.1 Exercises

1. Solve $x^2 - 9 = 0$ by the square root property.
2. Solve $x^2 = 20$.
3. Solve $(x-3)^2 = 16$.
4. Solve $x^2 - 5x + 6 = 0$ by factoring.
5. Solve $x^2 + 7x + 10 = 0$.
6. Solve $x^2 - 2x - 15 = 0$.
7. Solve $2x^2 + 5x - 3 = 0$.
8. Solve $x^2 - 6x + 9 = 0$ and state how many distinct roots there are.
9. Solve $x^2 + 4x + 1 = 0$ by completing the square.
10. Solve $x^2 - 2x - 2 = 0$ by the quadratic formula.
11. Solve $3x^2 - 2x - 1 = 0$.
12. Solve $x^2 + x + 1 = 0$ and interpret the result using the discriminant.
13. Find the zeros of $f(x) = x^2 - 4x$.
14. Use the discriminant to determine the number of real solutions of $4x^2 - 12x + 9 = 0$.
15. A ball is thrown upward; its height is $h(t) = -16t^2 + 32t + 48$ feet after t seconds. Find when it hits the ground ($h = 0$).
16. (☕) Without solving, use the discriminant to decide how many x-intercepts the parabola $y = x^2 + x + 5$ has, and explain what that means geometrically.
17. (☕) Explain why the quadratic formula is guaranteed to work for *every* quadratic, whereas factoring works only for some.
18. (☕) If a quadratic with integer coefficients has $x = 2 + \sqrt{3}$ as one solution, what must the other solution be, and why? (Hint: think about the $\pm$ in the formula.)

4.3 Analyzing Parabolas

The graph of every quadratic function is a *parabola*, and a handful of features, namely which way it opens, its axis of symmetry, its vertex, and where it crosses the axes, determine the entire shape. Reading these features lets us answer practical questions at a glance: the maximum height of a projectile is just the y-coordinate of the parabola's vertex, and the time it occurs is the vertex's x-coordinate.

Direction of opening. The sign of the leading coefficient a in $y = ax^2 + bx + c$ decides the direction: if $a > 0$ the parabola opens upward (vertex at the bottom, a minimum), and if $a < 0$ it opens downward (vertex at the top, a maximum). The magnitude of a controls how narrow the parabola is.



Positive a opens upward, negative a downward; larger $|a|$ gives a narrower parabola.

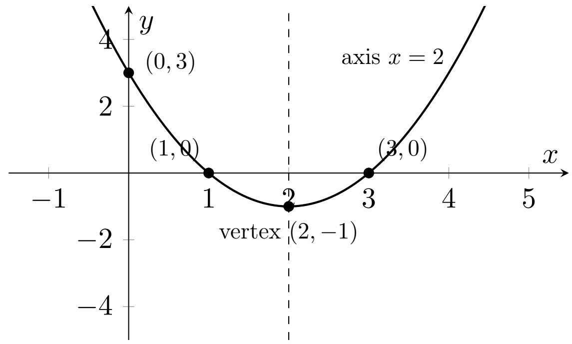
Axis of symmetry and vertex. Every parabola is symmetric about a vertical line, the *axis of symmetry*, given by

$$x = -\frac{b}{2a}.$$

The *vertex* is the point of the parabola that sits on this axis; its x-coordinate is $-\frac{b}{2a}$, and its y-coordinate is found by substituting that value back into f[1]. The vertex form $f(x) = a(x-h)^2 + k$ displays the vertex (h, k) directly.

Intercepts. The y-intercept is $f(0) = c$. The x-intercepts are the real zeros found by solving $ax^2 + bx + c = 0$; there are two, one, or none according to the discriminant.

[1] (☕) Why? What are we doing here?



The figure shows $y = x^2 - 4x + 3$: opening upward ($a = 1 > 0$), axis $x = -\frac{-4}{2} = 2$, vertex $(2, -1)$, x-intercepts 1 and 3, y-intercept 3.

Example 1. Analyze $f(x) = x^2 - 4x + 3$.

Since $a = 1 > 0$, it opens upward. Axis: $x = -\frac{-4}{2(1)} = 2$. Vertex y-value: $f(2) = 4 - 8 + 3 = -1$, so the vertex is $(2, -1)$, a minimum. Intercepts: $f(0) = 3$; and $x^2 - 4x + 3 = (x - 1)(x - 3) = 0$ gives x-intercepts 1 and 3.

Example 2 (Projectile). A rocket's height is $h(t) = -16t^2 + 64t + 80$ feet. Find its maximum height.

The vertex time is $t = -\frac{64}{2(-16)} = 2$ s. The maximum height is $h(2) = -16(4) + 64(2) + 80 = -64 + 128 + 80 = 144$ feet. (Because $a = -16 < 0$, the parabola opens downward, so the vertex is indeed a maximum.)

4.3.1 Exercises

1. State whether $y = 3x^2 - x + 1$ opens up or down.
2. State whether $y = -2x^2 + 5$ opens up or down.
3. Find the axis of symmetry of $y = x^2 - 6x + 2$.
4. Find the axis of symmetry of $y = 2x^2 + 8x - 1$.
5. Find the vertex of $y = x^2 - 2x - 3$.
6. Find the vertex of $y = -x^2 + 4x + 1$.
7. Find the y-intercept of $y = 5x^2 - 3x + 7$.
8. Find the x-intercepts of $y = x^2 - x - 6$.
9. Find the x-intercepts of $y = x^2 + 2x + 5$ (if any) and explain.
10. Write $y = x^2 - 6x + 5$ in vertex form by completing the square.
11. Find the minimum value of $f(x) = 2x^2 - 8x + 1$.

12. Find the maximum value of $f(x) = -x^2 + 6x - 5$.
13. Sketch $y = x^2 - 4$, labeling vertex and intercepts.
14. A farmer has 40 m of fence for a rectangular pen against a wall (three sides fenced). Express the area as a quadratic in the width and find the dimensions giving maximum area.
15. A ball's height is $h(t) = -16t^2 + 48t$ feet. Find the maximum height and the time it occurs.
16. (☕) Explain why the axis of symmetry passes exactly midway between the two x-intercepts when they exist, using the $\pm$ structure of the quadratic formula.
17. (☕) Without computing the vertex, explain how you know $f(x) = x^2 - 4x + 3$ and $g(x) = x^2 - 4x + 10$ share the same axis of symmetry.
18. (☕) A downward-opening parabola has its vertex below the x-axis. How many x-intercepts does it have, and why? Answer using only the direction and vertex location, no formula.

Chapter 5

Polynomial and Rational Functions and Inequalities

5.1 Polynomial Equations

In our last effort, we want to extend our study to higher-degree rational (and polynomial) equations and inequalities.

Fact. Any polynomial with real coefficients splits as a product of factors of degree 1 and 2, possibly with repetition.

This fact has an important consequence for us: if we are able to find roots of polynomials consistently, we can then apply the theory of linear and quadratic equations to find every possible real solution. Proving this fact is quite simple, but it requires concepts like continuity and limits, which are beyond the scope of this course. Also, observe that the irreducible degree 2 factors have to have negative discriminant, which we can compute directly form the coefficients.

Zeros and multiplicity. Recall that, by Ruffini's theorem, a real zero c of a polynomial corresponds to a factor $(x - c)$ in its prime decomposition. The number of times the factor appears is called its *multiplicity*; it is evident that a polynomial equation of degree n can have at most n solutions, counted with multiplicity.

Tackling the problem. How do we solve *in practice* a high-degree polynomial equation?
We must call on all our previously acquired skills in factoring polynomials, decompose the given polynomial into its lower degree factors, and then solve the simpler equations that we find.

An example. Solve $x^5 - 8x^4 + 18x^3 - 6x^2 - 19x + 14 = 0$.

Write $p(x) = x^5 - 8x^4 + 18x^3 - 6x^2 - 19x + 14$. The constant term is 14 and the leading coefficient is 1, so the rational root candidates are the divisors of 14: $\pm 1, \pm 2, \pm 7, \pm 14$. Testing them:

$$p(1) = 1 - 8 + 18 - 6 - 19 + 14 = 0.$$

So $x = 1$ is a root and $x - 1$ is a factor. Dividing $p(x)$ by $x - 1$ by long division we get

$$p(x) = (x - 1)(x^4 - 7x^3 + 11x^2 + 5x - 14).$$

We repeat the same procedure on the quotient $q(x) = x^4 - 7x^3 + 11x^2 + 5x - 14$. Its constant term is -14 and leading coefficient 1, so the same candidates $\pm 1, \pm 2, \pm 7, \pm 14$ apply. Testing:

$$q(-1) = 1 + 7 + 11 - 5 - 14 = 0,$$

so $x = -1$ is a root and $x + 1$ is a factor. Dividing $q(x)$ by $x + 1$ we get

$$p(x) = (x - 1)(x + 1)(x^3 - 8x^2 + 19x - 14).$$

Once more, on $r(x) = x^3 - 8x^2 + 19x - 14$: the constant term -14 and leading coefficient 1 again give candidates $\pm 1, \pm 2, \pm 7, \pm 14$. Testing:

$$r(2) = 8 - 32 + 38 - 14 = 0,$$

so $x = 2$ is a root and $x - 2$ is a factor. Dividing $r(x)$ by $x - 2$ we get

$$p(x) = (x - 1)(x + 1)(x - 2)(x^2 - 6x + 7).$$

We are left with the quadratic factor $x^2 - 6x + 7$, and by the zero-product property the original equation now reduces to $x - 1 = 0$, $x + 1 = 0$, $x - 2 = 0$, or $x^2 - 6x + 7 = 0$. The only remaining rational candidates for this last equation are the divisors of 7, namely $\pm 1, \pm 7$, and none of them is a root, so it has no rational roots at all, and so we apply the quadratic formula: with $a = 1,\ b = -6,\ c = 7$,

$$x = \frac{6 \pm \sqrt{36 - 28}}{2} = \frac{6 \pm \sqrt{8}}{2} = \frac{6 \pm 2\sqrt{2}}{2} = 3 \pm \sqrt{2}.$$

So the complete solution set of $p(x) = 0$ is

$$x = -1,\ 1,\ 2,\ 3 + \sqrt{2},\ 3 - \sqrt{2}.$$

Observe that since last two solutions are irrational, we would have never been able to find them other than with the quadratic formula.

5.1.1 Exercises

1. State the rational root candidates given by the rational roots test for $f(x) = x^3 + 2x^2 - 5x - 6$.
2. State the rational root candidates given by the rational roots test for $f(x) = 2x^3 + 3x^2 - 8x - 12$.

3. Verify that $x = -1$ is a root of $f(x) = x^3 + 2x^2 - 5x - 6$, and divide it out to find the remaining quadratic factor.
4. Solve completely: $x^3 + 2x^2 - 5x - 6 = 0$. (Use your work from the previous exercise.)
5. Solve completely: $x^3 - 4x^2 - 7x + 10 = 0$.
6. Solve completely: $x^3 - 3x^2 + 4 = 0$, stating the multiplicity of each solution.
7. Solve completely: $x^3 - 4x^2 + 2x + 4 = 0$.
8. Solve completely: $x^4 - 5x^3 + 5x^2 + 5x - 6 = 0$.
9. Find the zeros, with their multiplicities, of $f(x) = x^2(x - 3)$.
10. Find the zeros, with their multiplicities, of $f(x) = (x + 2)(x - 1)^2(x - 4)^3$.
11. Find a polynomial equation of least degree, with integer coefficients, whose solution set is $\{-2, 1, 3\}$.
12. Find a polynomial equation of least degree, with integer coefficients, having a solution of multiplicity 2 at $x = 3$ and a simple solution at $x = -1$.
13. What is the largest possible number of solutions, counted with multiplicity, of a degree-7 polynomial equation?
14. A degree-6 polynomial equation has real coefficients. Using the Fact, explain why it could have 0, 2, 4, or 6 real solutions counted with multiplicity, but never exactly 5.
15. (☕) The Fact states that every irreducible quadratic factor of a real polynomial must have negative discriminant. Explain why: what would a nonnegative discriminant tell you about that factor, and why would it contradict irreducibility?
16. (☕) Explain, using the Fact, why every polynomial equation of odd degree with real coefficients must have at least one real solution, while one of even degree might have none at all. (Think of the degree as a sum of 1's and 2's, one for each linear or irreducible quadratic factor.)
17. (☕) A polynomial equation $p(x) = 0$ of degree 5 with real coefficients has three rational solutions, $x = -1$, 2, 5, each of multiplicity 1, found via the rational roots test. After dividing out these three roots, what is the degree of the factor that remains, and what are the possible numbers of additional real solutions (counted with multiplicity) that $p(x) = 0$ could still have?

5.2 Polynomial and Rational Inequalities

Inequalities are mathematical “statements”[1] expressing a comparison between two objects. We will focus on polynomial and rational inequalities, which as can be easily inferred, compare two polynomials or two rational functions.
Solving an inequality means finding the numbers that we can evaluate the polynomials (or rational functions) on to make the resulting statement true. Differently from equations, solutions to inequalities usually come in the form of intervals, both open and closed.

The polynomial case. A polynomial inequality is a statement of the form $p(x) \geq 0$, where $p(x)$ is a polynomial. How do we solve a polynomial inequality? Everything rests on the observation that a polynomial function can change sign only at one of its zeros[2]. The first step is thus to find the roots of $p(x)$ (read: factor it completely), and “chop up” the real line into intervals on which the sign of p stays the same. Once we have this information, we can procede with one of the three possible methods exposed below.

First method: probing. Choose one point in each component, whichever is most convenient, and evaluate p there. Since the sign is constant on the whole component, the sign of that single value is the sign of p on the entire interval. This method works better when p has low degree, and the roots of the p are “well-behaved”, so that we can evaluate points without having to do many calculations.

Second method: the sign table. Factor p completely and study each factor separately, keeping in mind that the sign of a product is the product of the signs. A linear factor $ax - c$, with $a > 0$, is negative to the left of c and positive to its right, and conversely for $a < 0$; an irreducible quadratic factor $ax^2 + bx + c$, having negative discriminant, has no zeros whatsoever, and therefore keeps the sign of its leading coefficient a along the whole line. We record all of this in a table with one row per factor and one column per component, and read the sign of p off each column by multiplying down it. For $p(x) = (x+2)(x-1)$, whose zeros -2 and 1 cut the line into three components:

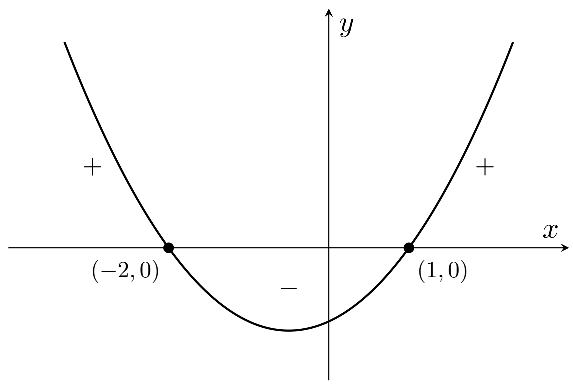
	$(-\infty, -2)$	$(-2, 1)$	$(1, +\infty)$
$x+2$	$-$	$+$	$+$
$x-1$	$-$	$-$	$+$
$(x+2)(x-1)$	$+$	$-$	$+$

Third method: parabolas. The degree 2 factors need not be probed at all, because we already know their graphs. The graph of $ax^2 + bx + c$ is a parabola, opening upward if $a > 0$ and downward if $a < 0$, and the discriminant tells us how it meets the axis. If the discriminant is positive there

[1]As with equations, they are more properly *formulas*, since they have free variables, but we call them statements drawing from the common usage of the word.

[2]This is due to *continuity*, a concept which is beyond the scope of this course.

are two distinct zeros, and the factor has the sign of a outside them and the opposite sign between them; if it is zero the parabola touches the axis at its vertex and has the sign of a everywhere else; if it is negative the parabola misses the axis entirely and has the sign of a on all of $\mathbb{R}$.



Whichever method we use, we still have to decide what happens *at* the zeros, and this is dictated by the inequality sign alone: a strict inequality ($<$ or $>$) excludes them, while $\leq$ and $\geq$ include them, with the exception of the zeros of the denominator in the rational case, which are always excluded because there the expression does not exist.

We now turn our attention to *rational* inequalities, which are statements of the form $\frac{p(x)}{q(x)} \geq 0$ (or > 0); in other words, we want to know when a given rational function is positive, and when it is negative. We are going to encounter both of them frequently, and so it would be useful to recall that the strict inequality, denoted by the symbols $<$ and $>$, exclude equality.
How do we go about solving these inequalities?

Reducing to the polynomial case. We observe first that $\frac{1}{q(x)}$ and $q(x)$ *always* have the same sign, since they have to multiply to $1 > 0$. Thus, the rational function $\frac{p(x)}{q(x)}$ is positive (or negative) if and only if the polynomial $p(x)q(x)$ is. Since we also want the fraction to exist, we have to impose that $q(x) \neq 0$. We can thus rephrase the inequality

$$\frac{p(x)}{q(x)} \geq 0$$

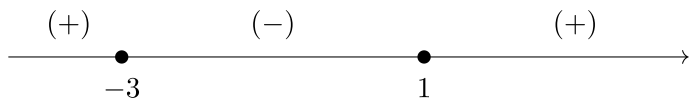
as

$$\begin{cases} p(x)q(x) \geq 0 \\ q(x) \neq 0 \end{cases} ;$$

solving the polynomial inequality as we said above will yield the solution we're after.

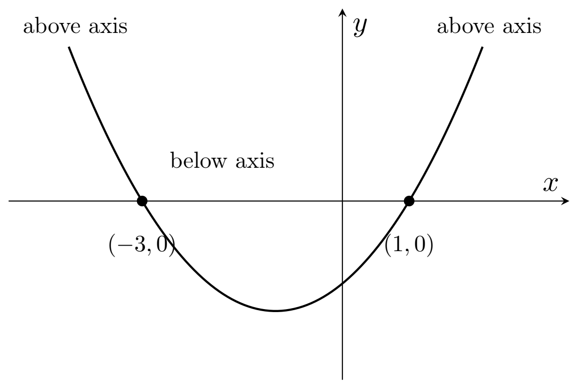
Example 1. Solve $x^2 + 2x - 3 > 0$.

By probing. Factor: $x^2 + 2x - 3 = (x + 3)(x - 1)$, with critical numbers -3 and 1. These split the real line into three intervals; testing a point in each tells us the sign of the product there.



The sign line above is for $(x+3)(x-1)$: positive outside the roots, negative between them. So $(x+3)(x-1) > 0$ on $(-\infty,-3) \cup (1,\infty)$.

By the parabola. The same inequality asks when $y = x^2 + 2x - 3$ lies strictly above the x-axis. Since $a = 1 > 0$, the parabola opens upward, with x-intercepts at $x = -3$ and $x = 1$ (the same critical numbers found above). Thus, the inequality is satisfied “outside" of the intercepts



Where the graph lies above the x-axis (outside the two roots), $y = x^2 + 2x - 3$ is positive; between the roots the graph dips below the axis. Reading this off the picture gives exactly the same solution as before: $(-\infty,-3) \cup (1,\infty)$.

Both methods, unsurprisingly, agree: they are two different ways of looking at the same fact.

Example 2. Solve $x^2 - x - 6 \geq 0$.

Factor: $(x-3)(x+2) \geq 0$, with critical numbers -2 and 3. Testing a point in each interval: at $x = -3$, $(-)(-) = +$; at $x = 0$, $(-)(+) = -$; at $x = 4$, $(+)(+) = +$. The expression is ≥ 0 on the outer intervals, and the endpoints satisfy “$= 0$,” so the solution is $(-\infty,-2] \cup [3,\infty)$.

Example 3 (Rational). Solve $\dfrac{x-1}{x+2} < 0$.

Critical numbers: $x = 1$ (numerator zero) and $x = -2$ (denominator zero, always excluded). Testing: at $x = -3$, $\frac{-}{-} = +$; at $x = 0$, $\frac{-}{+} = -$; at $x = 2$, $\frac{+}{+} = +$. The quotient is negative only on $(-2, 1)$. Both endpoints are excluded (-2 makes the denominator zero; 1 gives equality, not the strict $<$), so the solution is $x \in (-2, 1)$.

5.2.1 Exercises

1. Solve $x - 3 > 0$ and write the answer in interval notation.

2. Solve $(x-2)(x+1) > 0$.
3. Solve $(x-4)(x+3) < 0$.
4. Solve $x^2 - 9 < 0$.
5. Solve $x^2 - x - 6 \leq 0$.
6. Solve $x^2 + 2x - 8 > 0$.
7. Solve $x^2 \leq 5x$.
8. Solve $x(x-1)(x+2) > 0$.
9. Solve $(x-1)^2(x+3) < 0$.
10. Solve $\dfrac{1}{x} > 0$.
11. Solve $\dfrac{x-2}{x+1} > 0$.
12. Solve $\dfrac{x+4}{x-3} \leq 0$.
13. Solve $\dfrac{x}{x-2} \geq 0$.
14. Solve $\dfrac{x-1}{x^2-4} < 0$.
15. Solve $\dfrac{x^2-1}{x} \geq 0$.
16. (☕) Explain why an expression can change sign only at its zeros or where it is undefined, justifying why testing one point per interval is enough.
17. (☕) In Exercise 9 the factor $(x-1)^2$ never changes the sign. Explain why a squared factor can be ignored when determining sign (except at its own zero), and how it affects the solution set.
18. (☕) Contrast solving $\dfrac{x-2}{x+1} \geq 0$ with solving $(x-2)(x+1) \geq 0$. The critical numbers are the same, yet one endpoint is treated differently. Explain why, and identify which endpoint and the reason.

Appendix A

Complex Numbers

Throughout the course we have met, over and over, the same irritating sentence: *this equation has no solution.* This brief appendix is about what happens if we refuse to accept that sentence as final.

Solving equations. Asking to solve a polynomial equation is the most basic question one can ask of the arithmetic one already has. This is precisely what brought us from the naturals to the integers, and then to the rationals and reals. For example, the equation

$$x + 5 = 0$$

has no solution on the naturals, but it is perfectly meaningful, and refusing to answer it is a poor way of doing mathematics. So one *declares* that a solution exists, calls it -5, and checks that the enlarged collection still obeys the arithmetic rules the naturals obeyed. Repeating this process for all equations of the form

$$x + n = 0,$$

where n is any natural number, yields the integers, $\mathbb{Z}$. Similarly, we can ask to be able to solve all equations of the form

$$ax + b = 0,$$

where a and b are any integers, and we get the rationals, $\mathbb{Q}$. Passing to the reals is much more complicated, and so we do not deal with it.
However, given for granted the existence of real numbers, we know that some second-degree equations have no solution, as this is equivalent to having negative discriminant, and we can exhibit examples, such as

$$x^2 + 1 = 0.$$

In $\mathbb{R}$, the equation $x^2 + 1 = 0$, equivalent to $x^2 = -1$, has no solution, because the square of a real number is always nonnegative. Adjoining a solution produces the *complex numbers* $\mathbb{C}$, the subject of this appendix.

Constructing the complex numbers. We now exhibit a model of the complex numbers: let i be a number[1] with the property that $i^2 = -1$. We define the complex numbers as the set

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\},$$

equipped with the two operations

$$(a + bi) + (c + di) = (a + c) + (b + d)i, \qquad (a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i.$$

Observe that the real numbers are a subset of $\mathbb{C}$, realized by taking $\mathbb{R} = \{a + bi \in \mathbb{C} \mid b = 0\} \subseteq \mathbb{C}$.

Given a complex number $a + bi$, we say that the number a is its *real part*, written $\operatorname{Re} z$, and b is its *imaginary part*, written $\operatorname{Im} z$; note that the imaginary part is itself a *real* number. A complex number with $b = 0$ is called real, and one with $a = 0$ and $b \neq 0$ is called *purely imaginary*[2]. We now want to interpret equality in our new structure: since complex numbers are pairs, two of them are equal exactly when both coordinates agree:

$$a + bi = c + di \iff a = c \text{ and } b = d.$$

This concludes our definition of $\mathbb{C}$.

Square roots of negative numbers. For a positive real a, we set

$$\sqrt{-p} = i\sqrt{p},$$

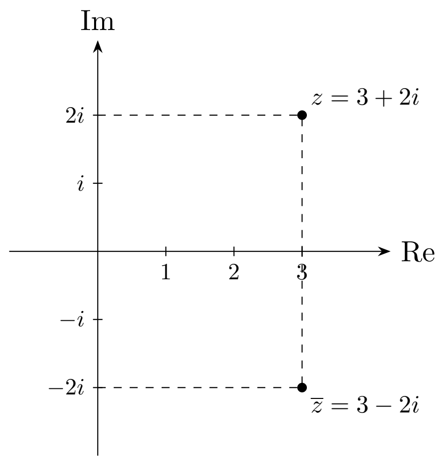
so that $\sqrt{-9} = 3i$ and $\sqrt{-50} = 5i\sqrt{2}$. A warning is in order: the rule $\sqrt{a}\sqrt{b} = \sqrt{ab}$, valid for nonnegative reals, is *false* for negative ones. Indeed $\sqrt{-1}\sqrt{-1} = i \cdot i = -1$, whereas $\sqrt{(-1)(-1)} = \sqrt{1} = 1$. Always convert to the form $i\sqrt{p}$ *first*, and only then multiply.

The ordering is lost. Extending the field of play has a big drawback, although different people will have different opinions on its severity. In $\mathbb{C}$, there is no ordering compatible with the arithmetic: there is no sensible way to say that one complex number is smaller than another. The reason is simple: in any ordered number system, a nonzero square must be positive; but $i^2 = -1$ would then force $-1 > 0$, while $1 = 1^2 > 0$ as well, and adding gives $0 > 0$. So the price of solving quadratics is the loss of the real line's ordering.

The complex plane. A complex number is a pair of real numbers, and a pair of real numbers is a point of the plane. Drawing $a + bi$ at the point (a, b) turns the plane into the *complex plane*, with the horizontal axis identified as the real numbers (it *is* the "old" real line) and the vertical axis being identified with the purely imaginary ones.

[1] The word "number" here means very little...

[2] The names here are an accident of history and unfortunate: there is nothing more imaginary about i than about -2 or $\sqrt{2}$, as these concepts are just as much of a human invention.



The complex plane with two points plotted.

Quadratic equations in $\mathbb{C}$. We can now go back to Section 4.2 and remove the exception that has been following us since.
Let $ax^2 + bx + c = 0$ with $a \neq 0$ and real coefficients, and suppose the discriminant is negative, say $b^2 - 4ac = -D$ with $D > 0$. Then $\sqrt{b^2 - 4ac} = i\sqrt{D}$, and the quadratic formula gives

$$x = \frac{-b \pm i\sqrt{D}}{2a}.$$

These are two complex numbers, and they are conjugate[3] to one another. Combining this with the two cases we already knew, the discriminant (which is real) can again be used to read the number of solutions: positive means two distinct real solutions, zero means one repeated real solution, negative means two conjugate (nonreal) complex solutions. *Every* quadratic equation with real coefficients has exactly two solutions in $\mathbb{C}$, counted with multiplicity, with no exceptions whatsoever.

Sums of squares factor. In Section 1.6 we noted that $a^2 + b^2$ does not factor over the reals. Over the complex numbers it does, and by the ordinary difference-of-squares pattern:

$$a^2 + b^2 = a^2 - (bi)^2 = (a + bi)(a - bi).$$

In particular $x^2 + 1 = (x + i)(x - i)$ and $x^2 + 9 = (x + 3i)(x - 3i)$. These two facts should shed some light on a particularly important concept: saying that "$x^2 + 4$ is irreducible" is a statement about the polynomial itself just as much as it is about the number system we want to live in.

Example 1. Solve $x^2 + 25 = 0$.

Directly, $x^2 = -25$, so $x = \pm\sqrt{-25} = \pm 5i$. Equivalently, $x^2 + 25 = (x + 5i)(x - 5i)$ and the zero-product property finishes the job.

[3] Two complex numbers are conjugates if their real part is equal and their imaginary parts only differ by a sign.

Example 2. Solve $x^2 - 2x + 5 = 0$.

Here $a = 1$, $b = -2$, $c = 5$, so the discriminant is $4 - 20 = -16 < 0$, and

$$x = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

Example 3. Solve $x^2 + x + 1 = 0$, the equation left unresolved in Exercise 12 of Section 4.2.

The discriminant is $1 - 4 = -3$, so

$$x = \frac{-1 \pm i\sqrt{3}}{2},$$

two conjugate solutions. These two numbers, together with 1, are the three solutions of $x^3 = 1$, since $x^3 - 1 = (x - 1)(x^2 + x + 1)$: the equation "$x$ cubed is one" turns out to have three solutions, and only one of them is visible on the real line.

The Fundamental Theorem of Algebra. At this point, the reader might worry: we had to define $\mathbb{C}$ to solve $x^2 + 1 = 0$; will we now have to invent something else to solve $x^4 + 1 = 0$, and something else again after that? The answer is the most remarkable fact in this appendix, and the reason the chain of extensions ends here.

Theorem (Argand,Gauss, 1799-1806)**.** *Every polynomial of positive degree with complex coefficients has a root in* $\mathbb{C}$.

Combining this with Ruffini's theorem from Section 1.5 and repeating the argument n times, we get the form in which we will use it: every polynomial of degree $n \geq 1$ with complex coefficients factors completely into degree one factors, so that a polynomial equation of degree n has *exactly* n solutions in $\mathbb{C}$, counted with multiplicity.

The beautiful theory of complex numbers has many interesting phenomena that greatly differentiate it from the theory of reals, and oftentimes changing perspective from the real to the complex realm trivializes proofs and questions in an almost perfect way, but we explore it no further, for now.